

Controlling Parent Systems Through Swarms Using Abstraction

Kyle L. Crandall, and Adam M. Wickenhieser

Abstract—This study considers the control of parent-child systems where a parent system is acted on by a set of child systems with their own inputs. Examples of such systems include a swarm of robots pushing an object over a surface, a swarm of aerial vehicles carrying a large load, or a set of end effectors manipulating an object. In this paper, a general framework for decoupling the swarm from the parent system through a low-dimensional abstract state space is presented. The requirements of this framework are presented along with how constraints on both systems propagate through the abstract state and impact the requirements of the controllers for both systems. Several controllers with hard state constraints are designed to track a desired angle of the tilting plane with a swarm of robots driving on top. Both homogeneous and heterogeneous swarms of varying sizes and properties are considered. The controllers are shown to be locally asymptotically stable and are demonstrated in simulation.

Index Terms—Swarms, Adaptive Control, Cascade Systems, Robust Control.

I. INTRODUCTION

THE appeal of using multiple, cooperating agents to perform complicated tasks has been clearly demonstrated in many applications [1]. For example, when compared to single robot systems, swarms can be designed to be more robust to failure of single nodes, and can take advantage of the inherent high degree of parallelism. They also tend to be more economical with regard to reusability, maintenance, and scalability.

We consider a specific class of swarm robots that use multiple agents to affect a parent dynamical system. One case that fits in this class is a swarm of ground robots seeking to push an object over a surface. A common approach to this problem is caging, or arranging the swarm around the object so that the object is constrained to move with the swarm. Some approaches assume unknown geometry and use the swarm to estimate it [2], others exploit a priori knowledge of the geometry of the object and desired trajectory to loosen the constraints of the caging [3]. Other approaches exploit this a priori knowledge to define a set of high level behaviors for the swarm that can be achieved in a decentralized manner [4]. Gross and Dorigo propose an artificial neural network based approach that links robots together if they need to exert more force than an individual can provide [5]. These caging approaches assume powerless objects, with relatively simple

and stable dynamics that are not taken into account in the control design. This precludes the evaluation of these methods' ability to handle disturbances to the parent.

Another object manipulation problem is using a swarm of multirotors to carry an object. Goodrazi and Lee present a proportional controller coupled with feed-forward terms for using a swarm of quadrotors to transport a rigid body. They take into account the dynamics of not just the object being transported but also the cable attaching it to the quadrotors [6]. Klausen et al. present a different approach by dividing the problem into two parts, path following and coordination [7]. Both of these approaches isolate the dynamics of an individual member of the swarm and determine how it needs to move, either with the rest of the swarm or relative to the parent system. However, Goodrazi and Lee propose the use of a centralized controller with full state feedback on both members of the swarm and the parent system consisting of the load and the cables supporting it. Klausen et al. propose a method similar to the caging methods in which motions are assigned to the swarm, and it is assumed that the load follows.

The class of system we are considering includes other situations that would not traditionally be considered swarm problems. An example of this is an array of control surfaces on the wing of an aircraft such as proposed by Blower et al. [8]. This array of flaps can be viewed as a "swarm" and the wing frame forms the "parent" system being controlled. Boberg et al. propose a similar type of system for performing active damping on a suspension bridge [9]. Both of these papers propose centralized approaches to solving their respective problems.

In this study, we propose a new control architecture that encapsulates the coupling between the dynamics of the parent and swarm systems with an abstract state, allowing two controllers to be designed for each system independent of the exact composition of the swarm system. Thus, if the composition of the swarm changes, only the mapping to the abstract space needs to be adjusted. A controller for the swarm can then be designed in the abstract space. When combined, these controllers allow the swarm to control the parent system. Such an approach presents several advantages: it takes into account the dynamics of both the parent and the swarm systems, decouples the problem into two subsystems that can be considered individually, and reduces the amount of information needed to be shared across the swarm. A consequence of this approach is a degree of modularity in the controller design as well as swarm composition. The controller for the parent system can be swapped out for a different one without affecting the controller for the swarm system and

KL. Crandall is with the Department of Mechanical and Aerospace Engineering, The George Washington University, Washington, DC, 20052 USA e-mail: crandallk@gwu.edu.

AM. Wickenhieser is with The George Washington University.

Manuscript received Feb. 23, 2017.

vice-versa. Furthermore, this framework is not restricted to homogeneous swarm systems. This means that as long as each member knows how it contributes to the abstraction of the swarm relative to the rest of the swarm, a controller can be designed for it. This abstraction enables the mixture and exchange of multiple types of robots in a swarm without effecting the higher-level control.

This paper formally defines an abstraction-based framework for parent-swarm systems and provides analysis of the interactions between the parent system and the swarm building on the work presented in [10]. Specifically, in this paper we analyze how constraints on the swarm's state and inputs translate into the abstract space, presenting additional constraints on the controllers. We also demonstrate the modularity of this architecture by designing several different controllers on an example system and present simulation results illustrating the utility of this abstract framework.

A. Summary

Our proposed framework abstracts the swarm of robots to simplify the interactions between the parent and child systems. Using this abstraction, the equations of motion of the system are rewritten as a cascading system, and a backstepping-like approach is taken to controlling the overall system. This architecture requires a parent controller that specifies a desired abstraction for the swarm and a swarm controller that achieves this desired condition. In section II, we formally define the problem and discuss the decoupling of the system using abstraction. We then discuss how constraints on both the swarm and the parent system map through the abstract state and how they should be taken into account when designing the controllers for the various parts of the system. Finally, we discuss how to analyze the stability of these systems. In section III we present an example case and apply this architecture to design controllers independently for the parent and swarm systems. We then present simulation results to validate these controllers and our general architecture in section IV.

II. DEFINITIONS AND PROBLEM FORMULATION

We propose an architecture that decouples the two subsystems of a parent-child system by finding an abstraction of the swarm's state and inputs that embody the interactions between these two systems. First we formally define the type of system we are working with.

Definition II.1 (Parent-Child System). *The parent-children system is a system which can be divided into to two subsystems, a "swarm" of children, and a parent system, where the inputs to the system directly effect the the swarm subsystem, but only influence the parent system through the swarm system.*

A. Swarm System

Definition II.2 (Child System). *A child system consists of at least one degree of freedom that is directly effected by one or more inputs.*

Arbitrary child i has a state $\mathbf{x}_{si} \in X_{si} \subset \mathbb{R}^{q_i}$ and an input $\mathbf{u}_{si} \in U_{si} = \mathbb{R}^{r_i}$. The state exists in the constrained domain

X_{si} , which represents admissible states. The input exists in the domain U_{si} . q_i and r_i are the dimensions of the state and input for child system i , respectively.

A child system is comprised of a set of degrees of freedom; however, these degrees of freedom are not necessarily independent from degrees of freedom in other children. Thus the child system is governed by the differential equation

$$\dot{\mathbf{x}}_{si} = f_{si}(\mathbf{x}_s, \mathbf{u}_{si}) \text{ where } f_{si} : X_s \times U_{si} \rightarrow TX_{si} \quad (1)$$

In this equation, $\mathbf{x}_s \in X_s = \prod_{i=1}^N X_{si}$. The T operator denotes the tangent bundle.

Definition II.3 (Swarm System). *The collection of all of the children in a system forms the swarm system.*

We consider a swarm of N children with a swarm state, \mathbf{x}_s and input $\mathbf{u}_s \in U_s = \prod_{i=1}^N U_{si}$. We can then define the dynamics of the swarm with the differential equation:

$$\dot{\mathbf{x}}_s = f_s(\mathbf{x}_s, \mathbf{u}_s), \text{ where } f_s : X_s \times U_s \rightarrow TX_s \quad (2)$$

Note that f_s is just the concatenation of all the f_{si} functions for $i \in 1 \dots N$.

B. Parent System

Definition II.4 (Parent System). *The parent system consists of the degrees of freedom of the full parent-child system that are not acted upon directly by the inputs to the system.*

The state of the parent system as given by $\mathbf{x}_p \in X_p \subset \mathbb{R}^n$, where X_p represents the set of admissible parent states. The dynamics of the system are governed by the differential equation

$$\dot{\mathbf{x}}_p = f_p(\mathbf{x}_p, \mathbf{x}_s, \mathbf{u}_s) \text{ where } f_p : X_p \times X_s \times U_s \rightarrow TX_p \quad (3)$$

C. Abstraction of the Swarm

The crux of our proposed architecture is thr idea of abstraction. This idea is most closely related to Belta and Kumar's work, in which they propose using abstraction to control formations of robots [11]. They find an abstract state of the swarm that represents the formation's overall structure, position, and orientation. Rather than specifying terms that define the geometric features of the swarm, we use terms that represent the interaction between the swarm and the parent system.

Definition II.5 (Abstraction of a Swarm with Respect to a Parent System). *An abstraction of the swarm is a mapping between the state of the swarm to an abstract state that encapsulates all the interactions between the swarm and the parent systems as described below in (6). The dimension of this abstract state should be independent of the number of children in the swarm.*

This abstraction consists of an abstract state $\mathbf{a} \in A \subset \mathbb{R}^m$, and a mapping from the swarm state to the abstract state given in (4). This mapping must be surjective, i.e. every abstract state must represent at least one swarm state:

$$\mathbf{a} = \phi(\mathbf{x}_s) \text{ where } \phi : X_s \rightarrow A \quad (4)$$

We further require the dynamics of the parent system to be rewritten as

$$\dot{\mathbf{x}}_p = \hat{f}_p(\mathbf{x}_p, \mathbf{a}, \dot{\mathbf{a}}) \text{ where } \hat{f}_p : X_p \times A \times TA \rightarrow TX_p \quad (5)$$

where f_p is a composition of \hat{f}_p , f_s and ϕ :

$$f_p(\mathbf{x}_p, \mathbf{x}_s, \mathbf{u}_s) = \hat{f}_p\left(\mathbf{x}_p, \phi(\mathbf{x}_s), \frac{\partial \phi(\mathbf{x}_s)}{\partial \mathbf{x}_s} f_s(\mathbf{x}_s, \mathbf{u}_s)\right) \quad (6)$$

Once an abstraction of the swarm has been found, we can design a control law for the parent system in which the abstract state is considered to be the input and the swarm is considered independent of the parent. This controller provides a desired abstract trajectory for the swarm. Note that the equation of motion for the parent system, \hat{f}_p , is potentially a function of both the abstract state and its derivative. This dependency can be compensated for directly in the controller, or it can be treated as an uncertain disturbance term.

D. Constraint Considerations for Controller Design

Before we can design controllers for the parent and child systems, we must consider the constraints on the systems. Because the abstract mapping is not necessarily injective, a given abstract state corresponds to a set of swarm states, some of which may lie outside X_s , the set of permissible swarm states. Because of this, we partition the set of abstract states into three disjoint sets: absolutely constrained abstract states, partially constrained abstract states, and unconstrained abstract states.

Definition II.6 (Absolutely Constrained Abstract State). *An abstract state is said to be absolutely constrained if the set it maps to in the swarm state space exists entirely within the constraints on the swarm. The set of all absolutely constrained abstract states is the largest set A_A such that $\phi^{-1}[A_A] \subseteq X_s$.*

Definition II.7 (Unconstrained Abstract State). *An abstract state is said to be unconstrained if the set it maps to in the swarm state violates at least one constraint on the swarm state at all points. The set of all unconstrained abstract states is the largest set A_U such that $\phi^{-1}[A_U] \subseteq X_s^c$.*

Definition II.8 (Partially Constrained Abstract State). *An abstract state is said to be partially constrained if the set it maps to in the swarm state violates any of the constraints on the swarm at some points, but not all points. The set of all partially constrained abstract states $A_P = A \setminus (A_A \cup A_U)$*

We choose a domain A_C on which we constrain the abstract state.

$$A_C \subseteq A_A \cup A_P \quad (7)$$

If $A_C \subseteq A_A$, the swarm controller does not have consider the swarm constraints since all configurations of the swarm that correspond to that abstract state satisfy the constraints on the swarm. If $A_C \cap A_P \neq \emptyset$, the swarm controller must ensure that it drives the swarm to a configuration that does not violate the constraints on the swarm.

E. Stability

Given Lyapunov functions for the parent system and the swarm system, we can derive conditions on the stability of the overall system. First, we consider the stability of systems without constraints.

Theorem 1. *Given the following system:*

$$\begin{bmatrix} \dot{\mathbf{z}}_p^T & \dot{\mathbf{y}}_p^T \end{bmatrix}^T = f'_p(\mathbf{z}_p, \mathbf{y}_p) \quad (8)$$

$$\begin{bmatrix} \dot{\mathbf{z}}_a^T & \dot{\mathbf{y}}_a^T \end{bmatrix}^T = f'_a(\mathbf{z}_a, \mathbf{y}_a) \quad (9)$$

let \mathbf{z}_p and \mathbf{y}_p be functions of \mathbf{x}_p , and let \mathbf{z}_a and \mathbf{y}_a be functions of \mathbf{a} . Let $V_p(\mathbf{z}_p, \mathbf{y}_p)$ and $V_a(\mathbf{z}_a, \mathbf{y}_a)$ be positive definite functions with derivatives that satisfy the following:

$$\dot{V}_p \leq \dot{V}'_p(\mathbf{z}_p) + \dot{V}_{pc}(\mathbf{z}_p, \mathbf{z}_a) \quad (10)$$

$$\dot{V}_a \leq \dot{V}'_a(\mathbf{z}_a) + \dot{V}_{ac}(\mathbf{z}_p, \mathbf{z}_a) \quad (11)$$

If \dot{V}'_p and \dot{V}'_a are negative definite and

$$\left| \dot{V}'_p + \dot{V}'_a \right| > \left| \dot{V}_{pc} + \dot{V}_{ac} \right| \quad (12)$$

on some domain $\mathbf{z}_p \in D_p \setminus \{0\}$, $\mathbf{z}_a \in D_a \setminus \{0\}$, and

$$\left| \dot{V}_{pc}(0, 0) + \dot{V}_{ac}(0, 0) \right| = 0 \quad (13)$$

Then the origin of this system is locally stable with a region of attraction of the form $[\mathbf{z}_p^T \ \mathbf{z}_a^T \ \mathbf{y}_p^T \ \mathbf{y}_a^T]^T \in S \subset D_p \times D_a \times \mathbb{R}^{\dim(\mathbf{y}_p)} \times \mathbb{R}^{\dim(\mathbf{y}_a)}$, where $\mathbf{z}_p \rightarrow 0$ and $\mathbf{z}_a \rightarrow 0$ as $t \rightarrow \infty$ and \mathbf{y}_p and \mathbf{y}_a remain bounded.

Proof. Let $V(\mathbf{z}_p, \mathbf{y}_p, \mathbf{z}_a, \mathbf{y}_a) = V_p(\mathbf{z}_p, \mathbf{y}_p) + V_a(\mathbf{z}_a, \mathbf{y}_a)$ be our Lyapunov candidate. Because V_p and V_a are positive definite, V is also positive definite.

We now take the derivative of V with respect to time and get

$$\dot{V} \leq \dot{V}'_p(\mathbf{z}_p) + \dot{V}'_a(\mathbf{z}_a) + \dot{V}_{pc}(\mathbf{z}_p, \mathbf{z}_a) + \dot{V}_{ac}(\mathbf{z}_p, \mathbf{z}_a) \quad (14)$$

$$\leq -\left| \dot{V}'_p(\mathbf{z}_p) + \dot{V}'_a(\mathbf{z}_a) \right| + \left| \dot{V}_{pc}(\mathbf{z}_p, \mathbf{z}_a) + \dot{V}_{ac}(\mathbf{z}_p, \mathbf{z}_a) \right| \quad (15)$$

Therefore, for \dot{V} to be negative semi-definite

$$\left| \dot{V}'_p + \dot{V}'_a \right| \geq \left| \dot{V}_{pc} + \dot{V}_{ac} \right| \quad (16)$$

Next, we find where $\dot{V} = 0$. This should only be true at the origin to satisfy the Lyapunov stability theorem (Theorem 4.1, [12]), therefore we get that, for $\mathbf{z}_p \neq 0$ and $\mathbf{z}_a \neq 0$

$$\left| \dot{V}'_p(\mathbf{z}_p) + \dot{V}'_a(\mathbf{z}_a) \right| \neq \left| \dot{V}_{pc}(\mathbf{z}_p, \mathbf{z}_a) + \dot{V}_{ac}(\mathbf{z}_p, \mathbf{z}_a) \right| \quad (17)$$

This combined with (16) give us the condition in (12).

Finally, we consider $\mathbf{z}_p = 0$ and $\mathbf{z}_a = 0$, which gives us (13). Given these conditions, we can say that \dot{V} is negative semi-definite on the domain $D = D_p \times D_a \times \mathbb{R}^{\dim(\mathbf{y}_p)} \times \mathbb{R}^{\dim(\mathbf{y}_a)}$. Thus, by Theorem 8.4 in [12], the terms \mathbf{z}_p and \mathbf{z}_a are asymptotically stable with the region of attraction

$$S = \{\mathbf{z} \in D \mid V(\mathbf{z}) \leq \min\{V(\mathbf{b}) \mid \mathbf{b} \in \partial D\}\} \quad (18)$$

while \mathbf{y}_p and \mathbf{y}_a remain bounded; thus, S is a closed neighborhood of the origin. \square

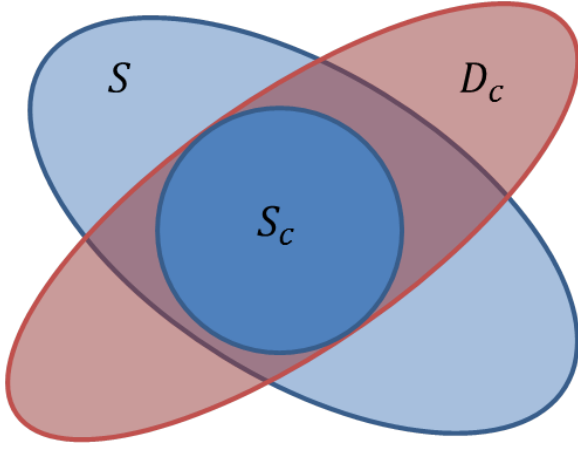


Fig. 1. Diagram of the sets defined in Theorems 1 and 2. S is the region of attraction of the origin of (8) and (9), D_c is the subset of additive perturbation vectors that do not violate any constraints, and S_c is the maximal set of contours of $V(\mathbf{z})$ contained within $S \cap D_c$.

Remark. Let $\phi(\mathbf{x}_s)$ be an abstraction of \mathbf{x}_s . If $\phi^{-1}[D_a]$ is bounded, where D_a is defined as in Theorem 1, then the swarm state is also bounded. However, if this inverse mapping from an abstract state is unbounded, then we cannot make any inference about the boundedness of the swarm state based on the boundedness of the abstract state.

We can now consider how the constraints on the system affect the region of attraction of the system:

Theorem 2. Consider a parent-swarm system with constrained spaces X_p and X_s for the parent and swarm states respectively, and let A_C be an absolutely constrained abstract space that contains the origin. Let $A_d \subseteq A_C$ be a surface on which the desired abstract state is further constrained. Let $\mathbf{z}_p(t)$ and $\mathbf{z}_a(t)$ be trajectories in the parent and abstract spaces that satisfy the conditions of Theorem 1. If there exist nonempty co-domains for the vectors \mathbf{z}_p and \mathbf{z}_a

$$D_{pc} = \{\mathbf{z}_p \in D_p \mid \mathbf{x}_{pd}(t) + \mathbf{b} \|\mathbf{z}_p\| \in X_p, \forall t \in [0, \infty), \mathbf{b} \in B^{dim(\mathbf{x}_p)}\} \quad (19)$$

$$D_{ac} = \{\mathbf{z}_a \in D_a \mid \mathbf{a}_d + \mathbf{b} \|\mathbf{z}_a\| \in A_C, \forall \mathbf{a}_d \in A_d, \mathbf{b} \in B^{dim(\mathbf{a}_d)}\} \quad (20)$$

where

$$B^n = \{\mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\| = 1\} \quad (21)$$

and $\mathbf{x}_{pd}(t)$ is the desired trajectory of the parent state, then there exists a nonempty domain S_c on which the origin of the system is asymptotically stable and D_{pc} and D_{ac} are forward time-invariant for all future time:

$$S_c = \{\mathbf{z} \in S \cap D_c \mid V_p(\mathbf{z}) + V_a(\mathbf{z}) \leq \min\{V_p(\mathbf{b}) + V_a(\mathbf{b}) \mid \mathbf{b} \in bd(S \cap D_c)\}\} \quad (22)$$

where $D_c = D_{pc} \times D_{ac} \times \mathbb{R}^{dim(\mathbf{y}_p)} \times \mathbb{R}^{dim(\mathbf{y}_p)}$

Proof. Let $\mathbf{z}_p \times \mathbf{z}_a \times \mathbf{y}_p \times \mathbf{y}_a \in S_c$. Then $\mathbf{z}_p \in D_{pc}$, and so $\mathbf{x}_{pd}(t) + \mathbf{b} \|\mathbf{z}_p\| \in X_p$. By Theorem 1, \mathbf{z}_p and \mathbf{z}_a converge to

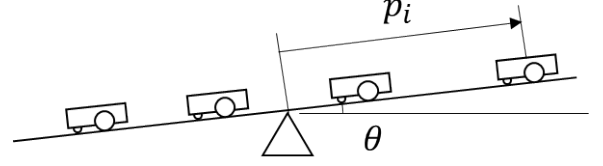


Fig. 2. Diagram of example system. The generalized coordinates are shown and labeled. θ is the angle the tilting plane makes with the ground, and p_i is the distance robot i is from the axis of rotation of the plane.

0 asymptotically with a region of attraction S . By Theorem 8.4 in [12], if $\mathbf{z}_0 \in S_c$, then $V(\mathbf{z}(t)) \rightarrow 0$ and hence $\mathbf{z}(t) \in S_c$ as $t \rightarrow \infty$. The relationships between these sets are illustrated in figure 1.

The sets D_{pc} and D_{ac} contain the origin since it is assumed that the desired trajectory of the parent system is feasible. S is the level set of a positive definite Lyapunov function; therefore, the intersection $S \cap D_c$ must contain the origin. The set S_c is also a level set of the same Lyapunov function, so it must also contain the origin. Therefore, as long as the desired trajectory and desired abstract surface satisfy all constraints, the set S_c at least contains the origin. \square

Using these two theorems, we can build a stability proof for the whole system based on the individual Lyapunov functions for the controllers for the parent and swarm systems. We can define a region of attraction that satisfies all constraints by mapping the constraints on the separate subsystems.

III. EXAMPLE CASE

To demonstrate our proposed controller architecture, we have chosen to apply it to a swarm of robots driving on top of a tilting plane. Our controller drives the robots to balance the plane at a desired angle or track a desired tilt angle trajectory. Figure 2 is a diagram of the proposed system. θ is the angle the surface of the tilting plane makes with the ground, and p_i is the position of robot i in the plane's coordinate system, where $p_i = 0$ at the axis of rotation of the plane. Robot i has a mass m_i .

In this example case, the swarm is the network of N robots driving on the plane; whereas the parent system is the plane itself, with a moment of inertia about its axis J and a Stribeck friction term on the axis of rotation $f_f(\dot{\theta})$. To examine the interactions between the parent and swarm systems, we state the equation of motion of the parent system:

$$\left(J + \sum_{i=1}^N m_i p_i^2 \right) \ddot{\theta} + g \cos(\theta) \sum_{i=1}^N m_i p_i + 2\dot{\theta} \sum_{i=1}^N m_i p_i \dot{p}_i + f_f(\dot{\theta}) = 0 \quad (23)$$

where g is the acceleration due to gravity.

Given these dynamics, we can identify the terms that embody the interaction between the parent system and the swarm. We choose the abstract state given in (24) and rewrite the dynamics of the parent system in terms of this state in

(25). In (25), τ_s and J_s represent the torque exerted on the level plane by the weight of the swarm and the moment of inertia of the swarm, respectively. This abstraction satisfies definition II.5 for a valid abstraction: the dimension of the abstract state is not dependent on the size of the swarm, and the abstract state encapsulates the interactions between the parent and swarm systems:

$$\mathbf{a} = \phi(\mathbf{x}_s) = \begin{bmatrix} \tau_s \\ J_s \end{bmatrix} = \begin{bmatrix} g \sum_{i=1}^N m_i p_i \\ \sum_{i=1}^N m_i p_i^2 \end{bmatrix} \quad (24)$$

$$\dot{\mathbf{x}}_p = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{\cos(\theta)\tau_s - \dot{\theta}J_s - f_f(\dot{\theta})}{J + J_s} \end{bmatrix} \quad (25)$$

To calculate the extent of the abstract space, we consider the mean and standard deviation of the swarm's mass distribution:

$$\mu_s = \frac{1}{m_t} \sum_{i=1}^N m_i p_i \quad (26)$$

$$\sigma_s^2 = \frac{1}{m_t} \sum_{i=1}^N m_i (p_i - \mu_s)^2 \quad (27)$$

We can show (28) by substituting (26) into the first element in the abstract mapping ϕ from (24) and applying the parallel axis theorem. We can then determine a lower bound of J_s for any given τ_s :

$$\tau_s = m_t g \mu_s \quad (28)$$

$$\begin{aligned} J_s &= m_t \sigma_s^2 + m_t \mu_s^2 \\ &= m_t \sigma_s^2 + \frac{1}{g^2 m_t} \tau_s^2 \\ &\geq \frac{1}{g^2 m_t} \tau_s^2 \end{aligned} \quad (29)$$

We now define the state equations for the swarm. We consider three different swarms: the first consists of robots that are one-dimensional single integrators controlled by velocity:

$$\mathbf{x}_s = \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix} \quad (30)$$

$$\dot{\mathbf{x}}_s = \mathbf{u}_s \quad (31)$$

The second swarm we consider consists of robots that are single-degree-of-freedom double integrators with passive linear damping controlled with a force input. This system has the same swarm state, but the equation of motion is

$$M\ddot{\mathbf{p}} + C\dot{\mathbf{p}} = \mathbf{u}_s \quad (32)$$

where M and C are diagonal matrices of the mass and linear damping coefficients respectively for each robot, and \mathbf{p}

is a vector of the positions of each robot in the swarm. In this case, the state representation of the swarm dynamics is

$$\dot{\mathbf{x}}_s = \begin{bmatrix} \dot{\mathbf{p}} \\ \ddot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -M^{-1}C \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \mathbf{u}_s \quad (33)$$

The third swarm is a non-homogeneous swarm consisting of members from both of the previous two swarms.

A. Constraint Analysis

Now that we have defined the abstraction of the system, we can consider the effects of constraints on both the parent and swarm system. First, there is a limit on how far the plane can tilt before the robots lose their traction and slip off the plane. We impose a limit on the angle of tilt in the plane resulting in the X_p defined in (34), where θ_{max} is the maximum tilt angle:

$$X_p = \{\mathbf{x}_p \in \mathbb{R}^2 \mid |\theta| \leq \theta_{max}\} \quad (34)$$

Concerning the child systems, we consider constraints on their position designed to prevent the robots from driving off the edge of the plane. These constraints generate the domain

$$X_{si} = \left\{ \mathbf{x}_{si} \in \mathbb{R} \mid |p_i| \leq \frac{L}{2} \right\} \quad (35)$$

We now find the mapping of these constrained spaces onto the abstract space. The constraints given in (35) describe an N -dimensional hypercube. We project this hypercube into the abstract space, which allows us to determine the bounds of A_C . When we map the vertices of this hypercube, we get a set of 2^N points at various values of τ_s but a constant value of J_s . When we map the edges, we get a set of curves connecting these points. We can use these boundaries in addition to the definitional constraint (29) to define the valid abstract space.

Furthermore, we need to describe the absolutely and partially constrained regions. We can see that $\phi^{-1}(\tau)$ is a hyperplane in the swarm space for each τ . Similarly, $\phi^{-1}(J_s)$ is a hyperellipsoid for each J_s . The surface that the abstract state maps to is the intersection of these two surfaces, e.g. swarm states that map to both parts of the abstract state. We can then check this surface against the constraints given in (35). Figure 3 illustrates these inverse mappings for a 3 robot swarm. Figure 4 shows a Monte Carlo approximation of the abstract spaces and the regions within.

This analysis of the regions of the abstract state that satisfy the various definitions of constraint, i.e. unconstrained, partially or absolutely constrained, can be used later to design the controller. By setting A_C to the absolutely constrained sectors, no special considerations need to be taken into account in the swarm controller design to prevent the swarm from violating its constraints.

B. Parent Controller

Given the constraints found in the previous section, we can now design a controller for the parent system. This controller must have a region of attraction contained within X_p . It must also control the system while ensuring that the desired

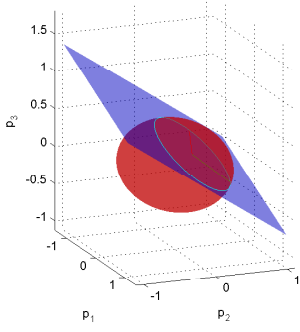


Fig. 3. Visualization of inverse mapping of an abstract state. The blue plane is all swarm states such that $\tau_s = 10N \cdot m$ and the red ellipsoid is all swarm such states such that $J_s = 1kg \cdot m^2$ for a three member swarm where $m_1 = 2kg$, $m_2 = 3kg$, and $m_3 = 3kg$. The cyan ellipse on the plane is the pre-image of $\mathbf{a} = [10 \ 1]^T$.

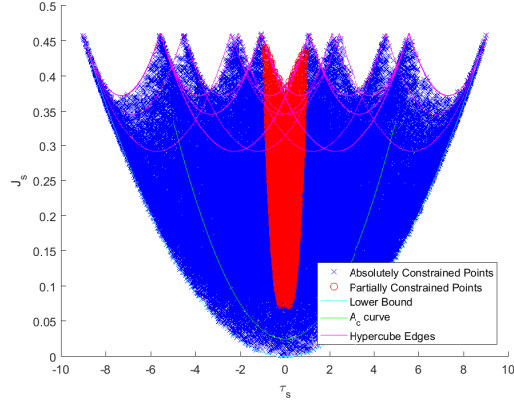


Fig. 4. Results of mapping the vertices and edges of the 4-dimensional hypercube into the abstract space. The magenta lines are the mappings of the edges of this hypercube. The cyan line is the lower bound of the abstract domain from (29). The blue Xs (absolutely constrained states) and red Os (partially constrained states) are the Monty Carlo mapping used to confirm the absolutely constrained region. The green curve is the surface A_c on which the desired abstract state is constrained. The 4 robots with masses between 0.25kg and 0.75kg and a plane length of 1m, which are the physical parameters of the system simulated in section IV

abstract state satisfies all of its constraints. Since we are only trying to control a single degree of freedom, θ , using both τ_s and J_s independently to control it would result in an over-actuated system. However, we can use Figure 4 to choose J_s to constrain the abstract state to avoid the partially constrained region (36). We further use this figure to choose τ_{max} (37) so that the curve represented in (36) stays within the absolutely constrained region.

$$J_{sd} = F_1 \tau_{sd}^2 + F_2 \quad (36)$$

$$|\tau_{sd}| \leq \tau_{max} \quad (37)$$

By constraining the desired abstract state to this curve within the absolutely constrained space, we ensure that the swarm does not violate its conditions so long as an envelope of sufficient size around this surface also satisfies the absolutely constrained conditions.

For the parent system, we apply two different controllers: a

PD controller and an Adaptive Robust Integral on the Sign of the Error (ARISE) controller based on the design proposed by Xian et al. [13] with a feed-forward adaptive term as proposed by Patre et al. [14][15].

The PD controller regulates the plane to a zero angle with zero velocity. Before designing the PD controller, we first adapt a linear friction model with a damping coefficient c and linearize the parent system about the origin:

$$\dot{\mathbf{x}}_p = \begin{bmatrix} 0 & 1 \\ 0 & \frac{c}{J+F_2} \end{bmatrix} \mathbf{x}_p + \begin{bmatrix} 0 \\ \frac{-1}{J+F_2} \end{bmatrix} \tau_{sd} \quad (38)$$

Next, we define the following control law:

$$\tau_{sd} = K_{pd} \mathbf{x}_p \quad (39)$$

where $K_{pd} \in \mathbb{R}^{1 \times 2}$ is the matrix of control gains. This matrix is set so that this system is a Linear Quadratic Regulator (LQR) that minimizes the following cost function:

$$\int_{t_0}^t \mathbf{x}_p^T Q \mathbf{x}_p(\tau) + R \tau_{sd}^2(\tau) d\tau \quad (40)$$

where Q is a symmetric, positive definite matrix and R is a positive scalar.

We also examine the ARISE controller, which can track trajectories bounded up to their fourth derivative in the presence of disturbances bounded up to their first derivative. We develop this controller from the equation of motion of the parent system

$$(J + J_s) \ddot{\theta} + J_s \dot{\theta} + f_f(\dot{\theta}) + \tau_d = -\cos(\theta) \tau_s \quad (41)$$

This equation uses the friction model

$$f_f(\dot{\theta}) = \gamma_1 \left(\tanh(\gamma_2 \dot{\theta}) - \tanh(\gamma_3 \dot{\theta}) \right) + \gamma_4 \tanh(\gamma_5 \dot{\theta}) + \gamma_6 \dot{\theta} \quad (42)$$

where γ_i are the parameters of the friction model, as well as a disturbance term τ_d .

The moment of inertia of the plane may be approximated based on measurements of the physical plane. The parameters of the friction model may be estimated using system identification techniques; however, we propose an adaptation law to estimate these terms. The uncertainties can be combined into a single term represented as a linear relationship between a vector of adaptive constants, λ , and a matrix that is a function of the desired state and its derivatives, Y_d :

$$Y_d \lambda = \begin{bmatrix} \ddot{\theta}_d \\ \tanh(\gamma_2 \dot{\theta}_d) - \tanh(\gamma_3 \dot{\theta}_d) \\ \tanh(\gamma_5 \dot{\theta}_d) \\ \dot{\theta}_d \end{bmatrix}^T \begin{bmatrix} J \\ \gamma_1 \\ \gamma_4 \\ \gamma_6 \end{bmatrix} \quad (43)$$

$$= J \ddot{\theta}_d + f_f(\dot{\theta}_d) \quad (44)$$

Using the error system developed in [13][14][15] given by $e_1 = \theta_d - \theta$ and $e_2 = \dot{e}_1 + \alpha_1 e_1$. We apply the control law given by

$$\tau_{sd} = -\sec(\theta) \left(Y_d \hat{\lambda} + (k_s + 1) e_2(t) - (k_s + 1) e_2(t_0) + \mu_1 \right) \quad (45)$$

$$\dot{\mu}_1 = (k_s + 1) \alpha_2 e_2(\sigma) + \beta \operatorname{sgn}(e_2(\sigma)) \quad (46)$$

$$\dot{\hat{\lambda}} = \hat{\lambda}(t_0) + \Gamma \dot{Y}_d^T e_2(\sigma) \Big|_{t_0}^t - \Gamma \mu_2 \quad (47)$$

$$\dot{\mu}_2 = \dot{Y}_d^T e_2(\sigma) - \alpha_2 \dot{Y}_d^T e_2(\sigma) \quad (48)$$

which utilize a measurement of e_1 and e_2 as well as an estimate of the unknown constants, $\hat{\lambda}$. The gains α_1 , α_2 , k_s , and β as well as the adaptive gain Γ must be sufficiently large to guarantee stability; however, these gains should not be so large that the system violates the bounds given in (37).

Note the secant term added to the controller in (45) to cancel out the cosine term in the dynamics of the parent system. This reduces the usable range of the controller to $\pm \frac{\pi}{2}$. However, the robots begin to slip off the plane at some $|\theta| < \frac{\pi}{2}$; thus, this domain is larger than the constraint given in (34).

C. Swarm Controller

The previous section derived a desired abstract state given by (36) and two parent control laws, a PD controller (39) and an ARISE controller (45)–(48). This section presents controllers for achieving these abstract states for the three different swarms.

1) *Single Integrator Swarm*: For the swarm described by (31), we use the input linearized controller proposed by Belta and Kumar [11]. This controller sets a control law on the derivative of the abstract state that is dependent on the errors in the abstract state given in (50). We first define the desired dynamics of the abstract state, where K is a diagonal, positive definite matrix and \mathbf{e}_a is the error in the abstract state:

$$\dot{\mathbf{a}} = K \mathbf{e}_a + \dot{\mathbf{a}}_d \quad (49)$$

$$\mathbf{e}_a = \mathbf{a} - \mathbf{a}_d = \begin{bmatrix} e_\tau \\ e_J \end{bmatrix} \quad (50)$$

If we take the time derivative of the abstract mapping (24) we get

$$\dot{\mathbf{a}} = \Phi_{xs} \dot{\mathbf{x}}_s \quad (51)$$

where Φ_{xs} is the Jacobian of the mapping function ϕ evaluated at the current \mathbf{x}_s . We combine (51) with the system model given in (31) to derive the control law given by

$$\mathbf{u}_s = \Phi_{xs}^\dagger (K \mathbf{e}_a + \dot{\mathbf{a}}_d) \quad (52)$$

The pseudo-inverse of the Jacobian can be written in terms of a set of parameters, S_0, \dots, S_3 , that are constant across the whole swarm:

$$\Phi_{xs}^\dagger = \frac{1}{S_3} \begin{bmatrix} m_1 S_2 - m_1 p_1 S_1 & \frac{1}{2} (m_1 p_1 S_0 - m_1 S_1) \\ \vdots & \vdots \\ m_N S_2 - m_N p_N S_1 & \frac{1}{2} (m_N p_N S_0 - m_N S_1) \end{bmatrix} \quad (53)$$

where

$$S_0 = \sum_{i=1}^N m_i^2 \quad (54)$$

$$S_1 = \sum_{i=1}^N m_i^2 p_i \quad (55)$$

$$S_2 = \sum_{i=1}^N m_i^2 p_i^2 \quad (56)$$

$$S_3 = \sum_{i=1}^N \sum_{j=i+1}^N m_i^2 m_j^2 (p_i - p_j)^2 \quad (57)$$

We can now write a control law for an individual child system:

$$u_{si} = \frac{1}{S_3} \left[\frac{1}{2} (m_i p_i S_0 - m_i S_1) \right]^T (K \mathbf{e}_a + \dot{\mathbf{a}}_d) \quad (58)$$

We demonstrate the stability of this controller with both parent controllers in the following theorems.

Theorem 3. *Given the error vector $\mathbf{z} = [\theta \ \dot{\theta} \ e_\tau \ e_J]^T$ The PD control law (39) on the parent system (25) and the single integrator control law (58) on the swarm (31) asymptotically stabilizes the origin of this system if*

$$k_1 > 0 \quad (59)$$

$$k_2 > \frac{j_{max}}{2 \cos(\theta_{max})} \quad (60)$$

$$K > 0 \quad (61)$$

$$\mathbf{z}_0 \in S \quad (62)$$

where k_1 and k_2 are the elements of the matrix K_{pd} in the parent controller, K is the controller gain for the swarm system, \mathbf{z}_0 is the initial condition of the system, and S is a region of attraction

$$S = \left\{ \mathbf{z} \in \mathbb{R}^4 \mid \|\mathbf{z}\|^2 \leq \frac{\theta_{max}^2}{2\eta} \right\} \quad (63)$$

where $\eta = \max \left\{ 1, \frac{J + F_1 \tau_{max}^2 + F_2}{2} \right\}$. The value θ_{max} is a limit on the angle θ such that $|\theta| \leq \theta_{max} < \frac{\pi}{2}$.

The proof of this theorem is provided in Appendix A.

Theorem 4. *Given the error vector $\mathbf{z} = [e_1 \ e_2 \ r \ e_\tau \ e_J]^T$ where $r = \dot{e}_2 + \alpha_2 e_2$, the ARISE control law (45) (46) (47), and (48) on the parent system (25) and the single integrator control law (58) on the swarm (31) asymptotically stabilizes the origin of this system if*

$$\alpha_1 > \frac{1}{2} \quad (64)$$

$$\alpha_2 > 1 \quad (65)$$

$$\beta > \zeta_{N_d} + \frac{1}{\alpha_2} \zeta_{\dot{N}_d} \quad (66)$$

$$k_s > \frac{c_{max}}{\eta_3} \quad (67)$$

$$K > 0 \quad (68)$$

$$\Gamma > 0 \quad (69)$$

$$\Gamma^T = \Gamma \quad (70)$$

where c_{max} is defined in Lemma 1 and

$$\eta_3 = \min \{2\alpha_1 - 1, \alpha_2 - 1, 1\} \quad (71)$$

The constants ζ_{N_d} and $\zeta_{\dot{N}_d}$ are defined in the proof. The desired trajectory satisfies $\theta_d, \dot{\theta}_d, \ddot{\theta}_d, \ddot{\theta}_d \in \mathcal{L}_\infty$ and $\tau_d, \dot{\tau}_d, \ddot{\tau}_d \in \mathcal{L}_\infty$.

The origin of this system has a region of attraction

$$S = \left\{ \mathbf{z} \in B, \mathbf{y}_p \in \mathbb{R}^2 \mid \left\| \begin{bmatrix} \mathbf{z}^T & \mathbf{y}_p^T \end{bmatrix} \right\|^2 \leq \frac{\eta_1}{\eta_2} \rho^2 \right\} \quad (72)$$

where $\rho = \rho_E^{-1} \left(2\eta_3 - \frac{c_{max}}{2k_s} + 2\lambda_{min} \right)$, λ_{min} is the smallest eigenvalue of K , and

$$\eta_1 = \frac{1}{2} \min \{1, J + F_2, \lambda_{min} (\Gamma^{-1})\} \quad (73)$$

$$\eta_2 = \frac{1}{2} \max \{2, J + F_1 \tau_{max}^2 + F_2, \lambda_{max} (\Gamma^{-1})\} \quad (74)$$

The proof for this theorem is provided in appendix B.

2) *Double Integrator Swarm*: For the swarm whose dynamics are given by (32), we have a second order system where the control input only directly effects the second derivative of the position. For this reason, we specify second-order dynamics for the abstract state:

$$\ddot{\mathbf{a}} = K_p \mathbf{e}_a + K_d \dot{\mathbf{e}}_a + \ddot{\mathbf{a}}_d \quad (75)$$

where K_p and K_d are diagonal, positive definite matrices.

We choose a control law for \mathbf{u}_s based on this equation:

$$\mathbf{u}_s = M \Phi^\dagger \left(K_p \mathbf{e}_a + (K_d - C_a) \dot{\mathbf{e}}_a - \dot{\Phi}_{xs} \dot{\mathbf{p}} + \ddot{\mathbf{a}}_d \right) + (C + k_{sd}) \Phi_{xs}^\dagger \dot{\mathbf{a}}_d - k_{sd} \dot{\mathbf{p}} \quad (76)$$

where k_{sd} is a positive scalar that adds damping to the individual members of the swarm, I is the identity matrix, and

$$C_a = \begin{bmatrix} C_{a11} & C_{a12} \\ C_{a21} & C_{a22} \end{bmatrix} \quad (77)$$

where

$$C_{a11} = \sum_{i=1}^N m_i (k_{sd} + c_i) (S_2 - p_i S_1) \quad (78)$$

$$C_{a12} = \frac{1}{2} \sum_{i=1}^N m_i (k_{sd} + c_i) (p_i S_0 - S_1) \quad (79)$$

$$C_{a21} = 2 \sum_{i=1}^N m_i p_i (k_{sd} + c_i) (S_2 - p_i S_1) \quad (80)$$

$$C_{a22} = \sum_{i=1}^N m_i p_i (k_{sd} + c_i) (p_i S_0 - S_1) \quad (81)$$

This feed-forward controller enforces the desired second-order dynamics in (75) on the abstract state, adds damping to each child system, and compensates the nonlinear effects of the damping in each child system. We can break this control law

for the whole swarm into a term for each individual child system:

$$u_{si} = \frac{1}{S_3} \left[\frac{1}{2} (m_i p_i S_0 - m_i S_1) \right]^T \left(m_i \left(K_p \mathbf{e}_a + (K_d - C_a) \dot{\mathbf{e}}_a - \dot{\Phi}_{xs} \dot{\mathbf{p}} + \ddot{\mathbf{a}}_d \right) + (c_i + k_{sdi}) \dot{\mathbf{a}}_d - k_{sdi} \dot{\mathbf{p}}_i \right) \quad (82)$$

Note that this control law, like the one given in (58), does not require knowledge of the other child systems' states, only the sums $S_0 \dots S_3$. We demonstrate the stability of this controller with both parent controllers in the following theorems.

Theorem 5. Given the error vector $\mathbf{z} = [\theta \ \dot{\theta} \ e_\tau \ e_J]^T$, the PD control law (39) on the parent system (25) and the double integrator control law (82) on the swarm (32) asymptotically stabilizes the origin of this system if the gains for the parent controller satisfy the same conditions on the parent system from Theorem 3 and if there exists some $\epsilon > 0$ such that the following are true:

$$\epsilon k_{pi} + k_{di} - \epsilon > 0 \quad (83)$$

$$\epsilon k_{pi} k_{di} - \epsilon^2 k_{pi} - \frac{\epsilon}{4} k_{di}^2 > 0 \quad (84)$$

$$k_{pi} + 1 > 0 \quad (85)$$

$$k_{pi} - \epsilon^2 > 0 \quad (86)$$

where k_1 and k_2 are the elements of the matrix K_{pd} in the parent controller, and k_{pi} and k_{di} are the i th diagonal elements of the K_p and K_d matrices respectively.

The origin is stable with the same region of attraction as given by Theorem 3.

The proof of this theorem is provided in Appendix C.

Theorem 6. Given the error vector $\mathbf{z} = [e_1 \ e_2 \ r \ e_\tau \ e_J]^T$ where $r = \dot{e}_2 + \alpha_2 e_2$, the ARISE control law (45) (46) (47), and (48) on the parent system (25) and the single integrator control law (82) on the swarm (32) asymptotically stabilizes the origin of this system if the gains for the controller satisfy the same conditions on the ARISE controller from Theorem 4 and there exists some $\epsilon > 0$ that satisfy the conditions given by Theorem 5.

This system has the same region of attraction as given by Theorem 4.

The proof for this theorem is provided in appendix D.

3) *Heterogeneous Swarm*: The final swarm we consider is a heterogeneous swarm consisting of single integrator members as defined in (31) as well as double integrator members as defined in (32). To control the swarm, we simply use the control laws from (58) and (82) for single integrator members and double integrator members respectively.

We demonstrate the stability of this controller with both parent controllers in the following theorems:

Theorem 7. Given the error vector $\mathbf{z} = [\theta \ \dot{\theta} \ e_\tau \ e_J]^T$, the PD control law (39) on the parent system (25) and the single integrator control law (58) for single integrator members of the swarm and the double integrator control law (82)

TABLE I
TABLE OF PHYSICAL PARAMETERS OF SIMULATED SYSTEM

Parameter	Value
m_{max}	$0.75kg$
m_{min}	$0.25kg$
c_{max}	$1.5N \cdot s/m$
c_{min}	$0.5N \cdot s/m$
J	$0.5kg \cdot m^2$
L	$1m$
γ_1	$0.01N$
γ_2	$1000s^2$
γ_3	$700s^2$
γ_4	$0.02N$
γ_5	$1000s^2$
γ_6	$1N\dot{s}$

for double integrator members of the swarm asymptotically stabilizes the origin of this system if the gains on the parent controller satisfy the same conditions as stated in Theorem 3, if $K > 0$ for the single integrator controllers, and there exists some $\epsilon > 0$ that satisfies the conditions given in Theorem 5.

The origin is stable with the same region of attraction as given by Theorem 3.

The proof of this theorem is provided in Appendix E.

Theorem 8. Given the error vector $\mathbf{z} = [e_1 \ e_2 \ r \ e_\tau \ e_J]^T$ where $r = \dot{e}_2 + \alpha_2 e_2$, the ARISE control law (45) (46) (47), and (48) on the parent system (25) and the single integrator control law (58) for single integrator members of the swarm and the double integrator control law (82) for double integrator members of the swarm asymptotically stabilizes the origin of this system if the gains on the parent controller satisfy the same conditions as stated in Theorem 4 and the controllers for the single and double integrator portions of the swarm satisfy the conditions given in Theorems 4 and 6 respectively.

This system has the same region of attraction as given by Theorem 4.

The proof for this theorem is provided in appendix F.

With these control laws, all an individual child system needs to know to implement its control law is the desired abstract state, the current abstract state, the four sums $S_0 \dots S_3$, the C_a matrix if it is a double integrator system, and its own state. It does not need to know the full state of the swarm nor even how big the swarm is. Further, this control law is independent of the size of the swarm.

IV. SIMULATIONS AND EXPERIMENTS

We demonstrate the example controllers developed in the previous section in simulation using different sized swarms. The physical parameters of the system are given in Table I. The masses of the child systems are uniformly distributed between m_{min} and m_{max} . The linear damping constants of the double integrator child systems are uniformly distributed between c_{min} and c_{max} .

For comparable results, we generated the masses and damping coefficients for the 4 child systems used in the following simulations. These values are $m_1 = 0.3552kg$, $m_2 = 0.3532kg$, $m_3 = 0.6762kg$, $m_4 = 0.4596kg$,

TABLE II
TABLE OF CONTROLLER PARAMETERS FOR PD

Parameter	Value
K	$10I$
K_p	$10I$
K_d	$5I$
k_{sd}	1
F_1	0.0125
F_2	0.025
τ_{max}	5

$c_1 = 0.7290Ns/m$, $c_2 = 1.4133Ns/m$, $c_3 = 0.6524Ns/m$, and $c_4 = 1.3258Ns/m$. These values for the masses of the robots were also used to generate Figure 4. These swarms also had a starting position of $\mathbf{p}_0 = [0.125m \ -0.125m \ 0.125m \ -0.125m]^T$ with an initial velocity of 0.

We apply the various controllers designed in section III to this system in a MATLAB simulation and present the results.

A. PD Controller

We implement the PD parent controller on the three different swarms with the gains given in Table II. These are LQR gains calculated using $Q = \text{diag}\{10, 1\}$ and $R = 1$. Each swarm consists of 4 robots with the mass, damping coefficients, and initial condition discussed previously. The initial angle of the plane is $0.1rad$ with 0 velocity. The controller gain for the parent system was calculated to be $K_{pd} = [3.1623 \ 3.2859]$.

The first swarm, consisting only of single integrator children, can be analyzed using Theorem 3. k_1 is positive, and K is a positive definite, symmetric matrix. We choose a $\theta_{max} = 0.2$ to ensure that the plane does not tilt so far as to cause the robots to slip off. We can now calculate the region of attraction to be when $\|\mathbf{z}\|^2 < 0.0326$. We then consider Figure 4 and determine acceptable constraints on our system to be $|e_J| \leq 0.03$ and $|e_\tau| < 2$. We can apply Theorem 2 to find that the constrained region of attraction to be when $\|\mathbf{z}\|^2 \leq 0.0326$. We can see that our initial errors $\mathbf{z}_0 = [0.1 \ 0 \ 0.0273 \ 0.0288]$ fall within this domain. We can calculate $\dot{J}_{max} = 1.9059$ and show that $k_2 > \frac{\dot{J}_{max}}{2 \cos(\theta_{max})}$, and thus all the conditions for stability are satisfied.

Figures 5 and 6 show the state and error plots respectively for the simulation of the first swarm. These plots show both the abstract error and the parent states converging to zero as guaranteed by the theorem.

The second swarm consists entirely of double integrator child systems. Based on theorem 5, this swarm has the same region of attraction and constrained region of attraction. The initial errors are also the same since the abstract state does not change with the swarm. Thus, the system's initial condition falls within the region of attraction.

Figures 7 and 8 show the state and error plots respectively for the simulation of the second swarm. Like the first swarm, the errors go to zero as expected; however, this swarm has a more clearly second order response with some overshoot. This is due to the poles of the linearization being complex.

The last swarm we consider is the heterogeneous swarm consisting of both single integrator and double integrator

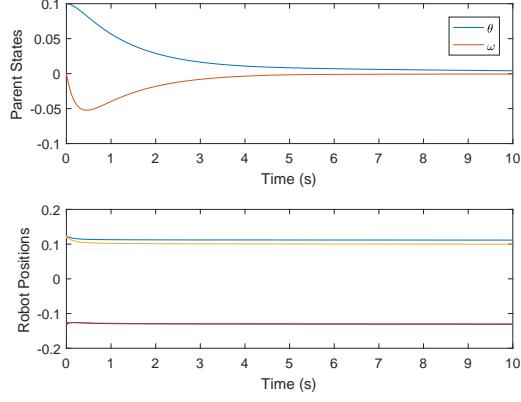


Fig. 5. State trajectories for parent and swarm subsystems with PD parent controller and single integrator swarm. The top plot is the parent state, the bottom plot is the swarm state

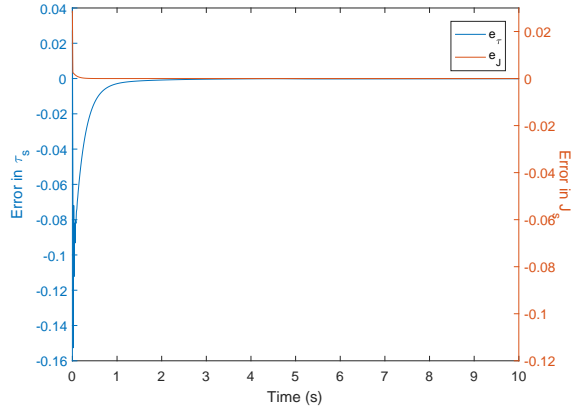


Fig. 6. Abstract error vs time plot for simulation of system with PD parent controller and single integrator swarm.

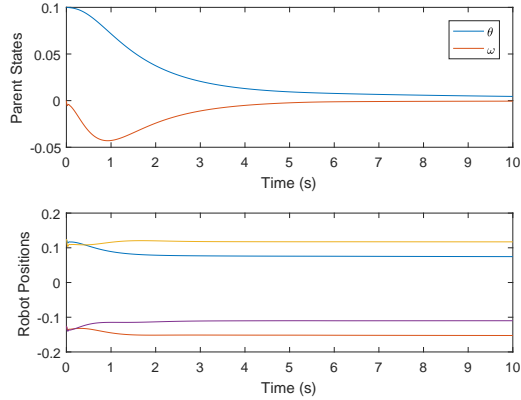


Fig. 7. State trajectories for parent and swarm subsystems with PD parent controller and double integrator swarm. Top plot is the parent state, the bottom plot is the swarm state

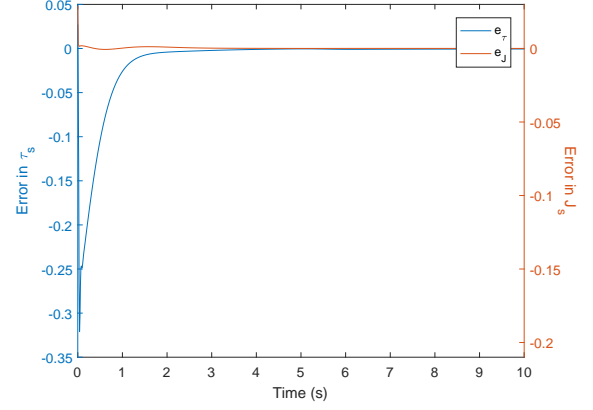


Fig. 8. Abstract error vs time plot for simulation of system with PD parent controller and double integrator swarm.

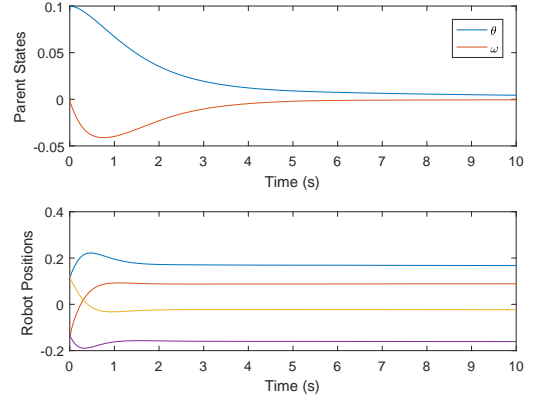


Fig. 9. State trajectories for parent and swarm subsystems with PD parent controller and a heterogeneous swarm. Top plot is the parent state, the bottom plot is the swarm state

robots. The first two members of the swarm are single integrators and the second two members are double integrators. These robots have the same masses and starting positions as the previous swarms. The double integrator robots have linear damping coefficients c_3 and c_4 . We can use Theorem 7 to show that the constrained region of attraction is the same as the last two and that the initial condition is within the region of attraction.

Figures 9 and 10 depict the results of this simulation. There is some oscillatory behavior from the swarm robots, but not as much as in the swarm of only double integrators. The poles of the linearization of the single integrators are real, and thus do not induce any oscillation.

B. ARISE Controller

The next set of simulations use the ARISE controller on the parent system with the three different swarms. The ARISE controller is implemented with the gains from Table III while the swarm system controllers use the gains from table II. For all of these experiments, the ARISE controller is attempting to track a reference trajectory $\theta_d(t) = 0.7 \sin(0.015\pi t)$ in the

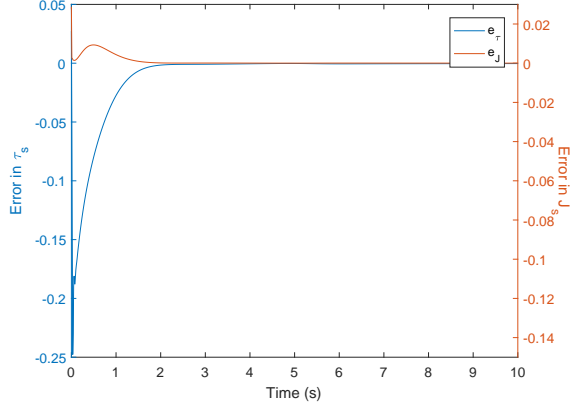


Fig. 10. Abstract error vs time plot for simulation of system with PD parent controller and heterogeneous swarm.

TABLE III
TABLE OF CONTROLLER GAINS FOR PD

Gain	Value
k_s	1
α_1	1
α_2	2
β	0.5
Γ	$\text{diag}(10, 1, 1, 10)$

presence of a sinusoidal disturbance $\tau_d(t) = 0.1 \sin(0.5t)$. The initial state of the plane is 0.075 rad with no velocity.

The first swarm tested is, again, the swarm made of single integrators. From Theorem 4 and the gains from Tables II and III, we can determine that $\eta_1 = 0.2625$, $\eta_2 = 10$, and $\eta_3 = 1$. With our chosen value of k_s , $c_{max} < 1$. The constant $\rho = 1.5$. Therefore, we can show that the region of attraction is when $\|z\| \leq 1.5$. Like with the PD controller, based on Figure 4, the constraint on the abstract error to stay in the absolutely constrained region is $|e_J| \leq 0.03$ and $|e_\tau| < 2$. We can then show that the constrained region of attraction is when $\|z\| \leq 0.0394$. Our initial states are $\mathbf{z}_{p0} = [-0.075 \ -0.0618 \ -0.1104]^T$, $\mathbf{y}_{p0} = [0.0434 \ 0.001 \ 0.01 \ 0.1]^T$, and $\mathbf{z}_{a0} = [0.0273 \ 0.0288]^T$. These initial conditions fall within this region of attraction; thus, the error system converges to 0 so long as c_{max} remains less than k_s . The k_s gain is tuned to result in a working controller based on expected values for c_{max} .

Figures 11 and 12 show the results of this simulation. The parent system tracks its desired trajectory well, even in the presence of sinusoidal noise. The swarm system quickly converges to its desired trajectory and follows it, allowing the ARISE controller to work on the parent system.

The second swarm the ARISE controller is applied to is the double integrator swarm. Like with the PD controller, the regions of attraction and initial conditions do not change with the new swarm; thus, the initial condition is within the region of attraction.

Figures 13 and 14 show the results of the simulation of this system. There is more oscillation in the swarm with this

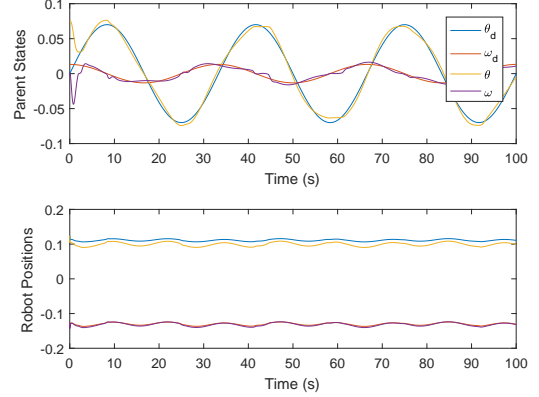


Fig. 11. State trajectories for parent and swarm subsystems with ARISE parent controller and a single integrator swarm. Top plot is the parent state, the bottom plot is the swarm state

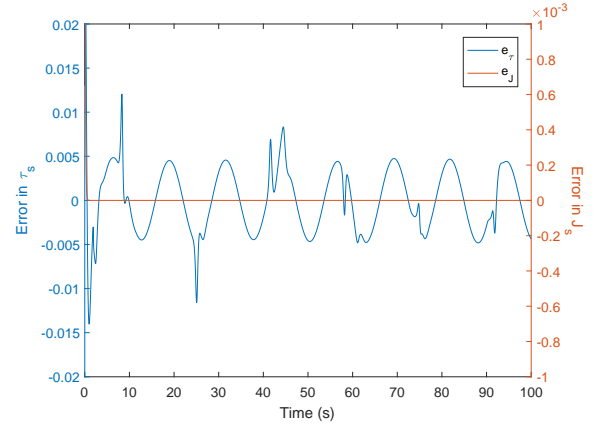


Fig. 12. Abstract error vs time plot for simulation of system with ARISE parent controller and single integrator swarm.

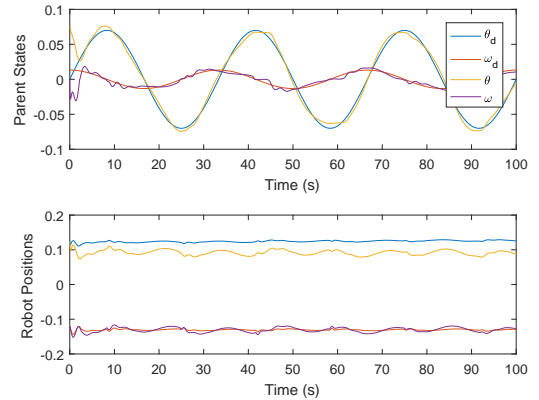


Fig. 13. State trajectories for parent and swarm subsystems with ARISE parent controller and a double integrator swarm. Top plot is the parent state, the bottom plot is the swarm state

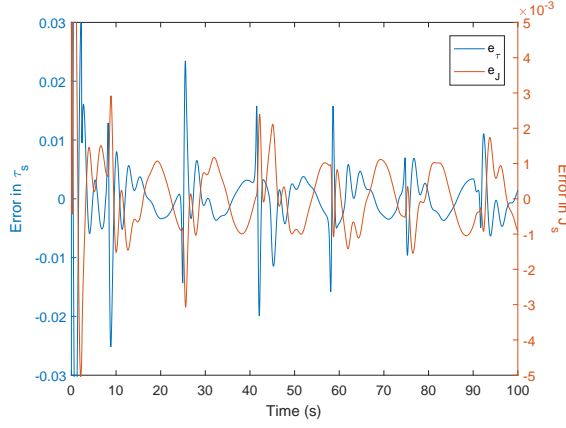


Fig. 14. Abstract error vs time plot for simulation of system with ARISE parent controller and double integrator swarm.

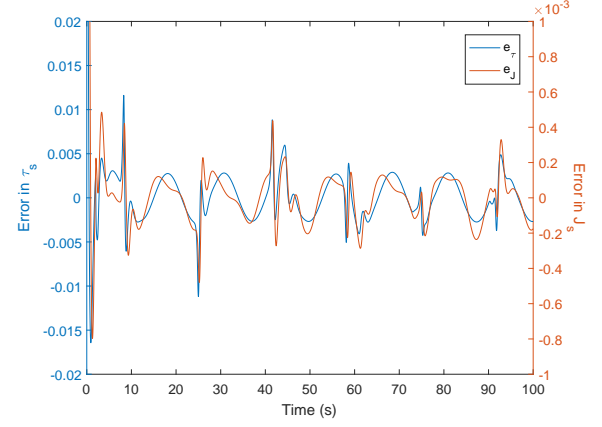


Fig. 16. Abstract error vs time plot for simulation of system with ARISE parent controller and heterogeneous swarm.

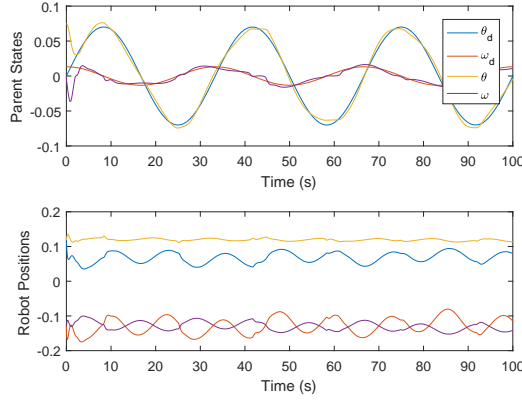


Fig. 15. State trajectories for parent and swarm subsystems with ARISE parent controller and a heterogeneous swarm. Top plot is the parent state, the bottom plot is the swarm state

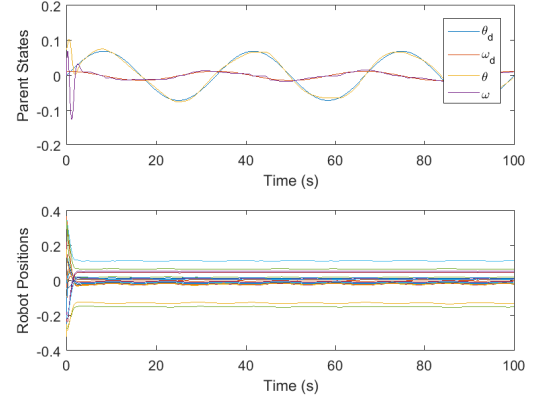


Fig. 17. State trajectories for parent and swarm subsystems with ARISE parent controller and a heterogeneous swarm of 10 single integrators and 10 double integrators. Top plot is the parent state, the bottom plot is the swarm state

system, but the tracking of the abstract state is very similar to that of the single integrator case, the only differences being some slight oscillation. this results in the trajectory of the parent system being nearly identical to that of the single integrator swarm.

The last swarm tested is the heterogeneous swarm. Again, similar to the PD controller, the regions of attraction are the same as for the other two swarms, as is the magnitude of the initial conditions.

Figures 15 and 16 show the results of this simulation. This swarm has a more oscillation than the single integrator swarm, but still not as much as the double integrator swarm. However, the tracking of the abstract state is better than with the double integrator swarm because half of the swarm only responded with first order dynamics. This can be seen especially in the J_{sd} signals in Figures 14 and 16. The end result however is that the tracking of the parent system being almost identical to the other swarms.

In addition to allowing different controllers to be used for both the parent and swarm systems, the abstraction is also independent of the size of the swarm. Figures 17, 18, 19, and 20 show the results for the ARISE controller on heterogeneous

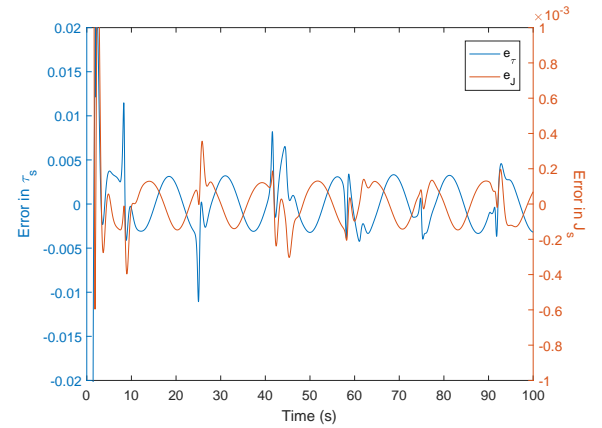


Fig. 18. Abstract error vs time plot for simulation of system with ARISE parent controller and heterogeneous swarm of 10 single integrators and 10 double integrators.

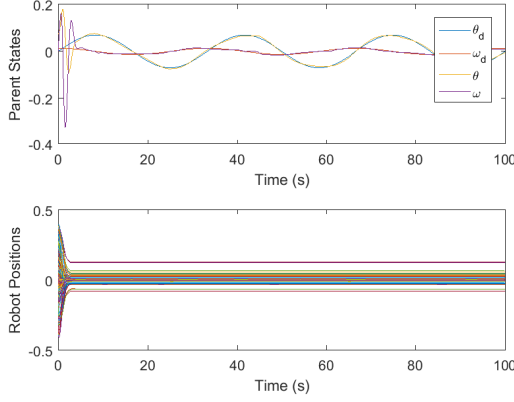


Fig. 19. State trajectories for parent and swarm subsystems with ARISE parent controller and a heterogeneous swarm of 100 single integrators and 100 double integrators. Top plot is the parent state, the bottom plot is the swarm state

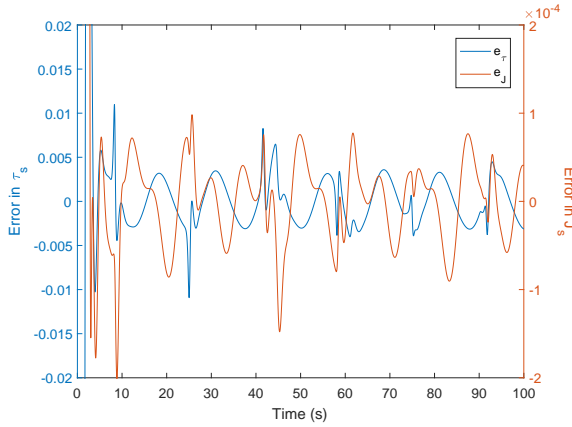


Fig. 20. Abstract error vs time plot for simulation of system with ARISE parent controller and heterogeneous swarm of 100 single integrators and 100 double integrators.

swarms of 20 and 200 robots respectively. The parent system response in these plots is very similar to the responses from the other example of the ARISE controller, the only change being the way the swarm moves. Note that the larger the swarm is, the tighter its final formation. This is because the controller is going to the same desired moment of inertia with more total mass. Also due to the larger mass, less movement is needed in the swarm to move the plane. This can be seen in the fact that the larger swarm's motions are less pronounced than those of the smaller swarms.

C. Results

The primary result of these simulations is the demonstration of the modularity of this architecture. We show that we can design different parent and swarm controllers and plug them into the architecture. While different parent controllers will give different results in the parent state, when we use different swarm controllers, or even completely different swarms, the response of the parent system does not change significantly. We can even combine different types of robots to form

heterogeneous swarms. The control for each robot is the same as if it were part of a homogeneous swarm.

These simulations also demonstrate that the size of the swarm does not significantly impact the performance of the parent controller. Changing the size of the swarm does change the swarm behavior but results in the same tracking of the abstract state, and thus the same tracking of the parent system.

It is also important to note the the swarm does not violate its assigned constraints. Because in each of these controllers we confine the abstract state to the absolutely constrained regions of the abstract space, the swarm is never in a position where any of its constraints are violated.

V. CONCLUSION

We propose a new architecture in which to design controllers for a class of systems where a swarm of subsystems acts upon a larger parent system with its own dynamics. This architecture condenses the interactions between the parent system into an abstract state of the swarm, which allows controllers for the parent system to be designed independent of the swarm. When designing controllers in this architecture, it is important to be aware of limitations on the swarm and the consequences these limitations have on the allowable abstract states that the parent controller can use.

We present an example case of a passive, dynamic parent system being manipulated by a large number of child subsystems. We propose an abstract state and design several controllers using our proposed architecture and demonstrate its inherent modularity. We then explore the structure of the abstract space in terms of the how an abstract state maps back to the swarm space and the constraints given for the swarm map into the abstract space. We exploit this structure to prevent the system from violating these constraints. This allows us to ensure that the swarm system will satisfy its constraints. Finally we demonstrate these controller in a MATLAB simulation to showcase the modularity of our architecture and its ability to work with swarms of various size and composition.

APPENDIX A PROOF OF THEOREM 3

Proof. To handle the friction in this case, we assume only the linear part of the friction term since we are building a linear controller. The rest of the equation of motion (25) is linearized about the origin. This results in the following linear function:

$$\dot{\mathbf{x}}_p = \begin{bmatrix} \dot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{\gamma_6}{J_s + F_2} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-1}{J_s + F_2} \end{bmatrix} \tau_{sd} = \mathbf{A}\mathbf{x}_p + \mathbf{B}\mathbf{u} \quad (87)$$

where $\mathbf{x}_p = [\theta \ \dot{\theta}]^T$ and $\mathbf{u} = \tau_s$. We can find the controllability matrix of this system:

$$\mathcal{C} = [\mathbf{B} \ \mathbf{A}\mathbf{B}] = \begin{bmatrix} 0 & \frac{-1}{J_s + F_2} \\ \frac{-\gamma_6}{(J_s + F_2)^2} & 0 \end{bmatrix} \quad (88)$$

The controllability matrix \mathcal{C} is full rank; thus, this system is controllable. We can find the control law (39) using an LQR method or pole placement, for example.

Let $V_p(\mathbf{z}_p)$ be a Lyapunov candidate for this system, where $\mathbf{z}_p = [\theta \quad \dot{\theta}]^T$.

$$V_p = \frac{1}{2}\dot{\theta}^2 + \frac{J + J_s}{2}\theta^2 \quad (89)$$

This function is positive definite with $\eta_1 \|\mathbf{z}_p\|^2 \leq V_p \leq \eta_2 \|\mathbf{z}_p\|^2$, where $\eta_1 = \min\{\frac{1}{2}, \frac{J}{2}\}$ and $\eta_2 = \max\{\frac{1}{2}, \frac{J+J_{s,max}}{2}\}$. where $J_{s,max}$ is some maximum value for J_s determined from the constraints on the abstract state given in (36) and (37).

Taking the time derivative of V_p and substituting (25) and (39):

$$\dot{V}_p = \theta\dot{\theta} + \dot{\theta}(J + J_s)\ddot{\theta} + \frac{J_s}{2}\dot{\theta}^2 \quad (90)$$

$$= \theta\dot{\theta} + \dot{\theta}\left(-\cos(\theta)\left(k_1\theta + k_2\dot{\theta}\right) - \dot{J}_s\dot{\theta} - f_f(\dot{\theta})\right) + \frac{J_s}{2}\dot{\theta}^2 \quad (91)$$

$$= -(\cos(\theta)k_1 - 1)\theta\dot{\theta} - \left(\cos(\theta)k_2 + \frac{J_s}{2}\right)\dot{\theta}^2 - \dot{\theta}f_f(\dot{\theta}) \quad (92)$$

$$= -\mathbf{z}_p^T \begin{bmatrix} 0 & -1 \\ \cos(\theta)k_1 & \cos(\theta)k_2 + \frac{J_s}{2} \end{bmatrix} \mathbf{z}_p - \dot{\theta}f_f(\dot{\theta}) \quad (93)$$

We assume that $|\theta| \leq \theta_{max} < \frac{\pi}{2}$ and by Lemma 4, for some domain $D_{J_{max}}$ on which $|\dot{J}_s| < \dot{J}_{max}$, thus

$$\dot{V}_p \leq -\mathbf{z}_p^T \begin{bmatrix} 0 & -1 \\ \cos(\theta_{max})k_1 & \cos(\theta_{max})k_2 - \frac{\dot{J}_{max}}{2} \end{bmatrix} \mathbf{z}_p - \dot{\theta}f_f(\dot{\theta}) = -\mathbf{z}_p^T M \mathbf{z}_p - \dot{\theta}f_f(\dot{\theta}) \quad (94)$$

We can show that the term $\dot{\theta}f_f(\dot{\theta})$ is positive definite since if $\dot{\theta} < 0$, then $f_f(\dot{\theta}) < 0$ and if $\dot{\theta} > 0$, then $f_f(\dot{\theta}) > 0$.

The $\mathbf{z}_p^T M \mathbf{z}_p$ term can be bounded by

$$\lambda_{min} \|\mathbf{z}_p\|^2 \leq \mathbf{z}_p^T M \mathbf{z}_p \leq \lambda_{max} \|\mathbf{z}_p\|^2 \quad (95)$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of M respectively. If these eigenvalues are both positive, then this term is positive definite. We find the characteristic equation of the matrix M :

$$\lambda^2 - \left(\cos(\theta_{max})k_2 - \frac{\dot{J}_{max}}{2}\right)\lambda + \cos(\theta_{max})k_1 = 0 \quad (96)$$

using Routh-Hurwitz Stability Criterion [16], the following need to be true for the eigenvalues to be positive:

$$\cos(\theta_{max})k_2 - \frac{\dot{J}_{max}}{2} > 0 \quad (97)$$

$$k_2 > \frac{\dot{J}_{max}}{2\cos(\theta_{max})} \quad (98)$$

$$\cos(\theta_{max})k_1 > 0 \quad (99)$$

$$k_1 > 0 \quad (100)$$

Therefore, if $k_1 > 0$ and $k_2 > \frac{\dot{J}_{max}}{2\cos(\theta_{max})}$, then \dot{V}_p is negative definite on the domain

$$D = \{\mathbf{z} \in D_{J_{max}} \mid |\theta| \leq \theta_{max}\} \quad (101)$$

We now consider the swarm system. We propose a quadratic Lyapunov function

$$V_a = \frac{1}{2}\mathbf{z}_a^T \mathbf{z}_a \quad (102)$$

where

$$\mathbf{z}_a = [e_\tau \quad e_J]^T \quad (103)$$

This function is clearly positive definite. We then take its time derivative to arrive at

$$\dot{V}_a = \dot{\mathbf{z}}_a^T \mathbf{z}_a \quad (104)$$

$$= (\dot{\mathbf{a}}_d - (K\mathbf{z}_a + \dot{\mathbf{a}}_d))^T \mathbf{z}_a \quad (105)$$

$$= -\mathbf{z}_a^T K^T \mathbf{z}_a \quad (106)$$

$$= \dot{V}'_a \quad (107)$$

Therefore, if K is positive definite, then \dot{V}'_a is negative definite on $D_a = \mathbb{R}^2$.

Define $V = V_p + V_a$. By Theorem 1, the domain on which \dot{V} is negative definite is

$$D = \{\mathbf{z} \in D_{J_{max}} \mid |\theta| \leq \theta_{max}\} \quad (108)$$

The largest level set that fits in this domain is

$$S_l = \left\{ \mathbf{z} \in \mathbb{R}^4 \mid V(\mathbf{z}) \leq \frac{\theta_{max}^2}{2} \right\} \quad (109)$$

For simplicity, we can find a the largest ball that fits into this level set:

$$S = \left\{ \mathbf{z} \in \mathbb{R}^4 \mid \|\mathbf{z}\| \leq \frac{\theta_{max}}{2\eta} \right\} \quad (110)$$

where $\eta = \max\left\{\frac{1}{2}, \frac{J+F_1\tau_{max}^2+F_2}{2}\right\}$. S is a subset of the region of attraction of this system. We can use this bound on $\|\mathbf{z}\|$ to find \dot{J}_{max} and thus the range of k_2 required to stabilize the origin of this system. \square

APPENDIX B PROOF OF THEOREM 4

Proof. We define a new error term r :

$$r = \dot{e}_2 + \alpha_2 e_2 \quad (111)$$

My multiplying r by the inertia term, we arrive at

$$(J + J_s)r = (J + J_s)(\ddot{e}_1 + \alpha_1 \dot{e}_1 + \alpha_2 e_2) \quad (112)$$

$$= (J + J_s)(\ddot{\theta}_d + \alpha_1 \dot{e}_1 + \alpha_2 e_2) - (J + J_s)\ddot{\theta} \quad (113)$$

$$= Y_d \lambda + W + \cos(\theta)\tau_s + \tau_d \quad (114)$$

where

$$W = (J + J_s)(\ddot{\theta}_d + \alpha_1 \dot{e}_1 + \alpha_2 e_2) + \dot{J}_s \dot{\theta} + f_f(\dot{\theta}) - Y_d \lambda \quad (115)$$

We now rewrite the control laws (45) and (47) in terms of r :

$$\tau_{sd} = -\sec(\theta)(Y_d \hat{\lambda} + \mu) \quad (116)$$

$$\dot{\lambda} = \Gamma \dot{Y}_d^T r \quad (117) \quad \text{and a Lyapunov candidate}$$

where

$$\mu(t_0) = 0 \quad (118)$$

$$\hat{\lambda}(t_0) = \hat{\lambda}_0 \quad (119)$$

$$\dot{\mu} = (ks + 1)r + \beta \text{sgn}(e_2) \quad (120)$$

and substitute (120), (50), and (116) into (114):

$$(J + J_s)r = Y_d \lambda + W - Y_d \hat{\lambda} - \mu - \cos(\theta) e_\tau + \tau_d \quad (121)$$

$$= Y_d \tilde{\lambda} + W - \mu - \cos(\theta) e_\tau + \dot{\tau}_d \quad (122)$$

where

$$\tilde{\lambda} = \lambda - \hat{\lambda} \quad (123)$$

We next take the time derivative of (122):

$$(J + J_s) \dot{r} = -\dot{J}_s r + \dot{Y}_d \tilde{\lambda} - Y_d \dot{\tilde{\lambda}} + \dot{W} - \dot{\mu} + \dot{\theta} \sin(\theta) e_\tau - \cos(\theta) \dot{e}_\tau + \dot{\tau}_d \quad (124)$$

and substitute (116) and (117):

$$(J + J_s) \dot{r} = -\dot{J}_s r + \dot{Y}_d \tilde{\lambda} - Y_d \Gamma \dot{Y}_d^T r + \dot{W} - (k_s + 1)r - \beta \text{sgn}(e_2) + \dot{\theta} \sin(\theta) - \cos(\theta) \dot{e}_\tau + \dot{\tau}_d \quad (125)$$

$$= -\frac{1}{2} \dot{J}_s r + \dot{Y}_d \tilde{\lambda} + N + E - (k_s + 1)r - \beta \text{sgn}(e_2) - e_2 \quad (126)$$

where

$$N = -Y_d \Gamma \dot{Y}_d^T r + \dot{W} - \frac{1}{2} \dot{J}_s r + e_2 + \tau_d \quad (127)$$

$$E = \dot{\theta} \sin(\theta) e_\tau - \cos(\theta) \dot{e}_\tau \quad (128)$$

Define:

$$\tilde{N} = N - N_d \quad (129)$$

$$\tilde{E} = E - E_d \quad (130)$$

and use Lemmas 1 and 2 to show

$$\|\tilde{N}\| \leq c_{max} \|\mathbf{z}_p\| \quad (131)$$

$$\|\tilde{E}\| \leq \rho_E (\|\mathbf{z}_p\|) \|\mathbf{z}_a\| \quad (132)$$

where

$$\mathbf{z}_p = [e_1 \quad e_2 \quad r]^T \quad (133)$$

$$\mathbf{z}_a = [e_\tau \quad e_J]^T \quad (134)$$

The last auxiliary functions we need are

$$P(t) = \beta |e_2(t_0)| - e_2(t_0) (N_d(t_0) + E_d(t_0)) - \int_{t_0}^t L(\sigma) d\sigma \quad (135)$$

$$L(t) = r((N_d(t) + E_d(t)) - \beta \text{sgn}(e_2(t))) \quad (136)$$

We now define a new vector \mathbf{y}_p that contains terms that are bounded but do not necessarily converge to zero, such as error in the adaptive terms and integral terms:

$$\mathbf{y}_p = [\tilde{\lambda}^T \quad \sqrt{P}]^T \quad (137)$$

$$V_p = e_1^2 + \frac{1}{2} e_2^2 + \frac{1}{2} (J + J_s) r^2 + P + \frac{1}{2} \tilde{\lambda}^T \Gamma^{-1} \tilde{\lambda} \quad (138)$$

By Lemma 3, $\int_{t_0}^t L(\sigma) d\sigma \leq \beta |e_2(t_0)| - e_d(t_0) (N_d(t_0) + E_d(t_0))$. Thus we can show that $P \geq 0$, thus $\sqrt{P} \in \mathbb{R}$ and

$$\eta_1 \left\| \begin{bmatrix} \mathbf{z}_p^T & \mathbf{y}_p^T \end{bmatrix}^T \right\| \leq V_p \leq \eta_2 \left\| \begin{bmatrix} \mathbf{z}_p^T & \mathbf{y}_p^T \end{bmatrix}^T \right\| \quad (139)$$

where

$$\eta_1 = \frac{1}{2} \min \{1, J + F_2, \lambda_{min}(\Gamma^{-1})\} \quad (140)$$

$$\eta_2 = \frac{1}{2} \max \{2, J + F_1 \tau_{max}^2 + F_2, \lambda_{max}(\Gamma^{-1})\} \quad (141)$$

We can show this by writing $V_p = \begin{bmatrix} \mathbf{z}_p^T & \mathbf{y}_p^T \end{bmatrix} Q \begin{bmatrix} \mathbf{z}_p^T & \mathbf{y}_p^T \end{bmatrix}^T$, where Q is a block diagonal matrix in which each of the coefficients form a block on the diagonal. All the blocks are 1-dimensional, except for Γ . η_1 and η_2 are the minimum and maximum eigenvalues of Q , which are positive if Γ is positive definite; thus our Lyapunov candidate is positive definite. Note also, that we arrive at condition (66) as a consequence of Lemma 3.

We then take the time derivative and substitute the control laws (45) and (47) to show that \dot{V}_p is negative semi-definite:

$$\dot{V}_p = 2e_1 \dot{e}_1 + e_2 \dot{e}_2 + (J + J_s) \dot{r} r + \frac{1}{2} \dot{J}_s r^2 + \dot{P} - \tilde{\lambda}^T \Gamma^{-1} \dot{\tilde{\lambda}} \quad (142)$$

$$= 2e_1 (e_2 - \alpha_1 e_1) + e_2 (r - \alpha_2 e_2) + -\frac{1}{2} \dot{J}_s r^2 + \dot{Y}_d \tilde{\lambda} r + (N + E) r - (k_s + 1) r^2 - \beta \text{sgn}(e_2) r - e_2 r + \frac{1}{2} \dot{J}_s r^2 - L - \tilde{\lambda}^T \Gamma^{-1} \Gamma \dot{Y}_d^T r \quad (143)$$

$$= -2\alpha_1 e_1^2 - \alpha_2 e_2^2 - (k_s + 1) r^2 + 2e_1 e_2 + (\tilde{N} + \tilde{E}) r + (N_d + E_d) r - \beta \text{sgn}(e_2) r - r(N_d + E_d - \beta \text{sgn}(e_2)) \quad (144)$$

$$= -2\alpha_1 e_1^2 - \alpha_2 e_2^2 - (k_s + 1) r^2 + 2e_1 e_2 + (\tilde{N} + \tilde{E}) r \quad (145)$$

$$\leq -(2\alpha_1 - 1) e_1^2 - (\alpha_2 - 1) e_2^2 - (k_s + 1) r^2 + (\tilde{N} + \tilde{E}) r \quad (146)$$

$$\leq -\eta_3 \|\mathbf{z}_p\|^2 - (k_s r^2 - c_{max} \|\mathbf{z}_p\| |r|) + \rho_E (\|\mathbf{z}_p\|) \|\mathbf{z}_a\| \|\mathbf{z}_p\| \quad (147)$$

where $\eta_3 = \min \{2\alpha_1 - 1, \alpha_2 - 1, 1\}$. Thus $\alpha_1 > \frac{1}{2}$ and $\alpha_2 > 1$ make η_3 positive and thus make \dot{V}_p' negative definite. We break this into its uncoupled and coupled terms per Theorem 1

$$V_p' = -\eta_3 \|\mathbf{z}_p\|^2 - (k_s r^2 - c_{max} \|\mathbf{z}_p\| |r|) \quad (148)$$

$$\dot{V}_{cp} = \rho_E (\|\mathbf{z}_p\|) \|\mathbf{z}_a\| \|\mathbf{z}_p\| \quad (149)$$

We complete the square on (147) to get

$$\dot{V}_p' \leq -\eta_3 \|\mathbf{z}_p\|^2 - \left(\left(\sqrt{k_s} |r| - \frac{c_{max} \|\mathbf{z}_p\|}{2\sqrt{k_s}} \right)^2 - \frac{c_{max}^2 \|\mathbf{z}_p\|^2}{4k_s} \right) \quad (150)$$

$$\leq -\eta_3 \|\mathbf{z}_p\|^2 + \frac{c_{max}^2 \|\mathbf{z}_p\|^2}{4k_s} - \left(\sqrt{k_s} |r| - \frac{c_{max} \|\mathbf{z}_p\|}{2\sqrt{k_s}} \right)^2 \quad (151)$$

$$\leq -\left(\eta_3 - \frac{c_{max}}{4k_s} \right) \|\mathbf{z}_p\|^2 \quad (152)$$

which is negative definite if $k_s > \frac{c_{max}}{4\eta_3}$ on $D_p = \mathbb{R}^3$.

We use the same Lyapunov function for the swarm from Theorem 3:

$$V_a = \frac{1}{2} \mathbf{z}_a^T \mathbf{z}_a \quad (153)$$

With these two Lyapunov functions, V_p and V_a , we can say that by Theorem 1, \mathbf{z}_p and \mathbf{z}_a asymptotically converge to the origin and \mathbf{y}_p is bounded, so long as

$$\left(\eta_3 - \frac{c_{max}}{4k_s} \right) \|\mathbf{z}_p\|^2 + \mathbf{z}_a^T K^T \mathbf{z}_a > \rho_E (\|\mathbf{z}_p\|) \|\mathbf{z}_p\| \|\mathbf{z}_a\| \quad (154)$$

Let $\mathbf{z}' = [\mathbf{z}_p^T \quad \mathbf{z}_a^T]^T$. We specify a domain D on which (154) is satisfied:

$$D = \left\{ \mathbf{z} \in D_p \times D_a \mid \left(\eta_3 - \frac{c_{max}}{4k_s} \right) \|\mathbf{z}_p\|^2 + \mathbf{z}_a^T K^T \mathbf{z}_a > \rho_E (\|\mathbf{z}_p\|) \|\mathbf{z}_p\| \|\mathbf{z}_a\| \right\} \quad (155)$$

By Theorem 1, we can find a region of attraction S in which the initial state must be for the system to converge.

We can see that

$$\left(\eta_3 - \frac{c_{max}}{4k_s} \right) \|\mathbf{z}_p\|^2 + \mathbf{z}_a^T K^T \mathbf{z}_a \geq \left(\eta_3 - \frac{c_{max}}{4k_s} \right) \|\mathbf{z}\|^2 + \lambda_{min} \|\mathbf{z}\| \quad (156)$$

$$\rho_E (\|\mathbf{z}_p\|) \|\mathbf{z}_p\| \|\mathbf{z}_a\| \leq \frac{1}{2} \rho_E (\|\mathbf{z}_p\|) \|\mathbf{z}\|^2 \quad (157)$$

where λ_{min} is the smallest eigenvalue of K . Thus we can find the largest ball B within the domain D :

$$B = \left\{ \mathbf{z} \in D \mid \|\mathbf{z}\| \leq \rho_E^{-1} \left(2\eta_3 - \frac{c_{max}}{2k_s} + 2\lambda_{min} \right) \right\} \quad (158)$$

The boundary of this ball is much easier to compute than the boundary of D ; thus, we can use its boundary to find a level set S that is contained within the region of attraction of this system:

$$S_l = \{ \mathbf{z} \in B, \mathbf{y}_p \in \mathbb{R}^2 \mid V_p(\mathbf{z}, \mathbf{y}_p) + V_a(\mathbf{z}) \leq \eta_1 \rho^2 \} \quad (159)$$

where $\rho = \rho_E^{-1} \left(2\eta_3 - \frac{c_{max}}{2k_s} + 2\lambda_{min} \right)$.

Like in the proof for Theorem 3, we can find the largest ball within this set to simplify our calculations later.

$$S = \left\{ \mathbf{z} \in B, \mathbf{y}_p \in \mathbb{R}^2 \mid \left\| [\mathbf{z}^T \quad \mathbf{y}_p^T]^T \right\| \leq \frac{\eta_1}{\eta_2} \rho^2 \right\} \quad (160)$$

S is a subset of the region of attraction of this system. \square

APPENDIX C PROOF OF THEOREM 5

Proof. First we propose a Lyapunov candidate for the swarm system:

$$V_a = \frac{1}{2} \mathbf{z}_a^T \begin{bmatrix} K_p & \epsilon I \\ \epsilon I & I \end{bmatrix} \mathbf{z}_a = \frac{1}{2} \mathbf{z}_a^T M_1 \mathbf{z}_a \quad (161)$$

where $\mathbf{z}_a = [\mathbf{e}_a^T \quad \dot{\mathbf{e}}_a^T]^T$.

We can say that $\lambda_{min} \|\mathbf{z}_a\|^2 \leq V_a \leq \lambda_{max} \|\mathbf{z}_a\|^2$ where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of M_1 respectively. Therefore, this function is positive definite if all the eigenvalues of M_1 are positive. We find the eigenvalues of M_1 can be found from

$$\det(M_1 - \lambda I) = \det((K_p - \lambda I)(1 - \lambda)I - \epsilon^2 I) \quad (162)$$

$$= \det(\lambda^2 I - (K_p + I)\lambda + K_p - \epsilon^2 I) \quad (163)$$

$$= (\lambda^2 - (k_{p1} + 1)\lambda + k_{p1} - \epsilon^2) (\lambda^2 - (k_{p2} + 1)\lambda + k_{p2} - \epsilon^2) \quad (164)$$

where k_{pi} is the i th diagonal element of K_p . From here we can see that to have positive eigenvalues, the following must be true:

$$k_{pi} - \epsilon^2 > 0 \quad (165)$$

for $i = \{1, 2\}$.

Next we consider the time derivative of the Lyapunov candidate:

$$\dot{V}_a = \mathbf{z}_a^T M_1 \dot{\mathbf{z}}_a \quad (166)$$

$$= \mathbf{e}_a^T K_p \dot{\mathbf{e}}_a + \epsilon \dot{\mathbf{e}}_a^T \ddot{\mathbf{e}}_a + \epsilon \dot{\mathbf{e}}_a^T \dot{\mathbf{e}}_a + \dot{\mathbf{e}}_a^T \ddot{\mathbf{e}}_a \quad (167)$$

We then substitute the swarm equation of motion (32) and control law (75):

$$\dot{V}_a = \mathbf{e}_a^T K_p \dot{\mathbf{e}}_a + \epsilon \dot{\mathbf{e}}_a^T \dot{\mathbf{e}}_a + (\epsilon \mathbf{e}_a^T + \dot{\mathbf{e}}_a^T) \left(\ddot{\mathbf{a}}_d - \dot{\Phi} \dot{\mathbf{p}} - \Phi M \left(M^{-1} \Phi^\dagger \left(K_p \mathbf{e}_a + (K_d - C_a) \dot{\mathbf{e}}_a - \dot{\Phi} \dot{\mathbf{p}} + \ddot{\mathbf{a}}_d \right) + (k_{sd} I + C) \Phi^\dagger \dot{\mathbf{a}}_d - (k_{sd} I + C) \dot{\mathbf{p}} \right) \right) \quad (168)$$

This expression can then be simplified to

$$\dot{V}_a = -\epsilon \mathbf{e}_a^T K_p \mathbf{e}_a - \epsilon \mathbf{e}_a^T K_d \dot{\mathbf{e}}_a - \dot{\mathbf{e}}_a^T (K_d - \epsilon I) \dot{\mathbf{e}}_a \quad (169)$$

$$= \mathbf{z}_a^T \begin{bmatrix} -\epsilon K_p & -\frac{\epsilon}{2} K_d \\ -\frac{\epsilon}{2} K_d & -K_d + \epsilon I \end{bmatrix} \mathbf{z}_a \quad (170)$$

$$= \mathbf{z}_a^T M_2 \mathbf{z}_a \quad (171)$$

Therefore, for \dot{V}_a to be negative, the eigenvalues of M_2 must be negative. We can find the eigenvalues of M_2 by finding

$\det(M_2 - \lambda I) = 0$:

$$0 = \det \left((\epsilon K_p + \lambda I) (K_d + (\lambda - \epsilon) I) - \frac{\epsilon^2}{4} K_d^2 \right) \quad (172)$$

$$= \det \left(\lambda^2 I + (\epsilon K_p + K_d - \epsilon I) \lambda + \epsilon K_p K_d - \frac{\epsilon^2}{4} K_d^2 \right) \quad (173)$$

$$= \left(\lambda^2 + (\epsilon k_{p1} + k_{d1} - \epsilon) \lambda + \epsilon k_{p1} k_{d1} - \frac{\epsilon^2}{4} k_{d1}^2 \right) \left(\lambda^2 + (\epsilon k_{p2} + k_{d2} - \epsilon) \lambda + \epsilon k_{p2} k_{d2} - \frac{\epsilon^2}{4} k_{d2}^2 \right) \quad (174)$$

where k_{di} is the i th diagonal element of K_d . From here, we can see that to get negative real parts for the eigenvalues, the following conditions must be met:

$$\epsilon k_{pi} + k_{di} - \epsilon > 0 \quad (175)$$

$$\epsilon k_{pi} k_{di} - \frac{\epsilon^2}{4} k_{di}^2 > 0 \quad (176)$$

for $i = \{1, 2\}$.

For this Lyapunov candidate, $\dot{V}_a \leq \dot{V}'_a = \lambda_{2max} \|\mathbf{z}_a\|$ and has no coupling terms.

We can use the Lyapunov candidate V_p from the proof of Theorem 3. If $k_1 > 0$, $k_2 > \frac{j_{max}}{\cos(\theta_{max})}$, and there exists some ϵ that satisfies (83), (84), (85), and (86), then by Theorem 1, there exists a Lyapunov candidate for the combined system $V = V_p + V_a$ that proves the origin of the combined system is locally asymptotically stable on a region of attraction

$$S = \left\{ \mathbf{z} \in \mathbb{R}^6 \mid V(\mathbf{z}) \leq \frac{\theta_{max}^2}{2} \right\} \quad (177)$$

where $\mathbf{z} = [\mathbf{z}_p^T \quad \mathbf{z}_a^T]^T$. \square

APPENDIX D PROOF OF THEOREM 6

Proof. We can use the same Lyapunov candidate for the swarm system from the proof for Theorem 5:

$$V_a = \frac{1}{2} \mathbf{z}_a^T \begin{bmatrix} K_p & \epsilon I \\ \epsilon I & I \end{bmatrix} \mathbf{z}_a = \mathbf{z}_a^T M_1 \mathbf{z}_a \quad (178)$$

We can show that this is positive definite and its derivative is negative definite if

$$\epsilon k_{pi} + k_{di} - \epsilon > 0 \quad (179)$$

$$\epsilon k_{pi} k_{di} - \frac{\epsilon^2}{4} k_{di}^2 > 0 \quad (180)$$

$$k_{pi} + 1 > 0 \quad (181)$$

$$k_{pi} - \epsilon^2 > 0 \quad (182)$$

We can use the Lyapunov candidate V_p from the proof of Theorem 4 for the parent system. If $\alpha_1 > \frac{1}{2}$, $\alpha_2 > 1$, $\beta > \zeta_{Nd} + \frac{1}{\alpha_2} \zeta_{dotNd}$, $k_s > \frac{c_{max}}{4\eta_3}$, $\Gamma > 0$, $\Gamma^T = \Gamma$, and there exists some ϵ that satisfies (83), (84), (85), and (86), then by Theorem 1, there exists a Lyapunov candidate for the combined system $V = V_p + V_a$ that proves the origin of the combined system

is locally stable where \mathbf{z}_p and \mathbf{z}_a goes to zero while \mathbf{y}_p will remain bounded on a region of attraction

$$S = \left\{ \mathbf{z} \in \mathbb{R}^9 \mid \|\mathbf{z}\|^2 \leq \frac{\eta_1}{\eta_2} \rho^2 \right\} \quad (183)$$

where $\mathbf{z} = [\mathbf{z}_p^T \quad \mathbf{y}_p^T \quad \mathbf{z}_a^T]^T$ and η_1 , η_2 , and ρ come from the Theorem 4. \square

APPENDIX E PROOF OF THEOREM 7

Proof. We break the swarm into two parts, the single integrator swarm with a state z_{as} , and the double integrator swarm z_{ad} . We choose the Lyapunov candidate $V_a = V_{as} + V_{ad}$ where $V_{as} = \frac{1}{2} \mathbf{z}_{as}^T \mathbf{z}_{as}$ from the proof for Theorem 3. $V_{ad} = \frac{1}{2} \mathbf{z}_{ad}^T M_1 \mathbf{z}_{ad}$ from the proof for Theorem 5. We can use the same proofs from these theorems to show that V_a is positive definite under the given conditions.

We can show that \dot{V}_a is negative definite because $\dot{V}_a = \dot{V}_{as} + \dot{V}_{sd}$ and we can show these two terms are negative definite using the proofs from the same theorems as above. Also note that neither term has a coupling term, so $\dot{V}_{ac} = 0$ for this swarm as well.

We can use the Lyapunov candidate V_p from the proof of Theorem 3. If $k_1 > 0$, $k_2 > \frac{j_{max}}{\cos(\theta_{max})}$, and there exists some ϵ that satisfies (83), (84), (85), and (86), then by Theorem 1, there exists a Lyapunov candidate for the combined system $V = V_p + V_a$ that proves the origin of the combined system is locally asymptotically stable on a region of attraction

$$S = \left\{ \mathbf{z} \in \mathbb{R}^6 \mid V(\mathbf{z}) \leq \frac{\theta_{max}^2}{2} \right\} \quad (184)$$

where $\mathbf{z} = [\mathbf{z}_p^T \quad \mathbf{z}_a^T]^T$. \square

APPENDIX F PROOF OF THEOREM 8

Proof. We can use the same Lyapunov candidate V_a from the proof for Theorem 7 and the Lyapunov candidate V_p from the proof for Theorem B. If $\alpha_1 > \frac{1}{2}$, $\alpha_2 > 1$, $\beta > \zeta_{Nd} + \frac{1}{\alpha_2} \zeta_{dotNd}$, $k_s > \frac{c_{max}}{4\eta_3}$, $\Gamma > 0$, $\Gamma^T = \Gamma$, and there exists some ϵ that satisfies (83), (84), (85), and (86), then by Theorem 1, there exists a Lyapunov candidate for the combined system $V = V_p + V_a$ that proves the origin of the combined system is locally stable where \mathbf{z}_p and \mathbf{z}_a goes to zero while \mathbf{y}_p will remain bounded on a region of attraction

$$S = \left\{ \mathbf{z} \in \mathbb{R}^9 \mid \|\mathbf{z}\|^2 \leq \frac{\eta_1}{\eta_2} \rho^2 \right\} \quad (185)$$

where $\mathbf{z} = [\mathbf{z}_p^T \quad \mathbf{y}_p^T \quad \mathbf{z}_a^T]^T$ and η_1 , η_2 , and ρ are defined in the proof of Theorem 4. \square

APPENDIX G SUPPORTING LEMMAS

Lemma 1. *Given the following:*

$$N = -Y_d \Gamma \dot{Y}_d^T r + \dot{W} - \frac{1}{2} \dot{J}_s r + e_2 + \tau_d \quad (186)$$

$$N_d = \dot{J}_s \ddot{\theta}_d + (J + J_s) \ddot{\theta}_d + \ddot{J}_s \dot{\theta}_d + \dot{J}_s \ddot{\theta}_d + \dot{f}_f \left(\dot{\theta}_d \right) - \dot{Y}_d \lambda + \dot{\tau}_d \quad (187)$$

we can show that

$$\left| \tilde{N} \right| \leq c_{max} \|\mathbf{z}\| \quad (188)$$

where

$$\mathbf{z} = [e_1 \quad e_2 \quad r \quad e_\tau \quad e_J] \quad (189)$$

$$\tilde{N} = N - N_d \quad (190)$$

Proof. We start by expanding N and substituting in any auxillary equations:

$$\begin{aligned} N = & -Y_d \Gamma \dot{Y}_d r + \dot{J}_s \ddot{\theta}_d + \dot{J}_s (\alpha_1 \dot{e}_1 + \alpha_2 \dot{e}_2) + (J + J_s) \ddot{\theta}_d + \\ & (J + J_s) (\alpha_1 \ddot{e}_1 + \alpha_2 \ddot{e}_2) + \ddot{J}_s \dot{\theta}_d - \ddot{J}_s \dot{e}_1 + \\ & \dot{J}_s \ddot{\theta}_d - \dot{J}_s \ddot{e}_1 + \dot{f}_f \left(\dot{\theta} \right) - \dot{Y}_d \lambda - \frac{1}{2} \dot{J}_s r + e_2 + \dot{\tau}_d \end{aligned} \quad (191)$$

Then we consider the definition of \tilde{N} and use the definitions of e_1 , e_2 , and r to put in terms of these variables.

$$\tilde{N} = N - N_d \quad (192)$$

$$\begin{aligned} = & -Y_d \Gamma \dot{Y}_d r + \dot{J}_s (\alpha_1 \dot{e}_1 + \alpha_2 \dot{e}_2) + \\ & (J + J_s) (\alpha_1 \ddot{e}_1 - \alpha_2 \ddot{e}_2) - \ddot{J}_s \dot{e}_1 - \dot{J}_s \ddot{e}_1 + \end{aligned} \quad (193)$$

$$\begin{aligned} & \dot{f}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right) - \frac{1}{2} \dot{J}_s r + e_2 \\ = & \left(-\frac{1}{2} \dot{J}_s - Y_d \Gamma \dot{Y}_d \right) r - \alpha_2 \left(1 + \dot{J}_s \alpha_2 \right) e_2 + \\ & (J + J_s) \dot{e}_2 + \left(\dot{J}_s \alpha_1 - \ddot{J}_s \right) \dot{e}_1 + \end{aligned} \quad (194)$$

$$\begin{aligned} & \left((J + J_s) \alpha_1 - \dot{J}_s \right) \ddot{e}_1 + \dot{f}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right) \\ = & \left(-\frac{1}{2} \dot{J}_s - Y_d \Gamma \dot{Y}_d \right) r + \left(1 + \dot{J}_s \alpha_2 \right) e_2 - \\ & \alpha_2 (J + J_s) (r - \alpha_2 e_2) + \left(\dot{J}_s \alpha_1 - \ddot{J}_s \right) (e_2 - \alpha_1 e_1) + \\ & \left((J + J_s) \alpha_1 - \dot{J}_s \right) (r - \alpha_1 (e_2 - \alpha_1 e_1) - \alpha_2 e_2) + \\ & \dot{f}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right) \end{aligned} \quad (195)$$

$$\begin{aligned} = & \left(-\frac{1}{2} \dot{J}_s - Y_d \Gamma \dot{Y}_d + (J + J_s) (\alpha_1 - \alpha_2) - \dot{J}_s \right) r + \\ & \left(1 + \left(2\dot{J}_s + (J + J_s) (\alpha_2 - \alpha_1) \right) \alpha_2 + J_s \alpha_1 - \right. \\ & \left. \ddot{J}_s - (J + J_s) \alpha_1^2 + \dot{J}_s \alpha_1 \right) e_2 + \\ & \left(\ddot{J}_s \alpha_1 - 2\dot{J}_s \alpha_1^2 + (J + J_s) \alpha_1^3 \right) e_1 + \dot{f}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right) \end{aligned} \quad (196)$$

We can find limits on J_s and its derivatives based on our chosen limits on τ_{sd} and its derivatives based on the relation given in (36):

$$J_s \leq J_{max} = F_1 \tau_{max}^2 + F_2 \quad (197)$$

$$\left| \dot{J}_s \right| \leq \dot{J}_{max} = 2F_1 |\dot{\tau}_{max} \tau_{max}| \quad (198)$$

$$\left| \ddot{J}_s \right| \leq \ddot{J}_{max} = 2F_1 (|\ddot{\tau}_{max} \tau_{max}| + \dot{\tau}_{max}^2) \quad (199)$$

where τ_{max} , $\dot{\tau}_{max}$, and $\ddot{\tau}_{max}$ are the limits on τ_{sd} and its first and second derivative respectively. With these limits, we can say that

$$\begin{aligned} \left| \tilde{N} \right| \leq & \left(\frac{1}{2} \dot{J}_{max} + \left| Y_d \Gamma \dot{Y}_d \right| + (J + J_{max}) |\alpha_1 - \alpha_2| + \dot{J}_{max} \right) |r| + \\ & \left| 1 + \left(2\dot{J}_{max} + (J + J_{max}) (\alpha_2 - \alpha_1) \right) \alpha_2 + J_{max} \alpha_1 \right| |e_2| + \\ & \left| \ddot{J}_{max} + J \alpha_1^2 + \dot{J}_{max} \alpha_1 \right| |e_2| + \\ & \left(\ddot{J}_{max} \alpha_1 + 2\dot{J}_{max} \alpha_1^2 + \left(J + \dot{J}_{max} \right) \alpha_1^3 \right) |e_1| + \\ & \left| \dot{f}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right) \right| \end{aligned} \quad (200)$$

We now consider specifically the term $\dot{f}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right)$. For the parameters that are not adaptive, we use $\bar{\gamma}_i$ to represent the estimate of the value of the parameter, then (42) gives

$$\begin{aligned} \bar{f}_f \left(\dot{\theta} \right) - f_f \left(\dot{\theta}_d \right) = & \gamma_1 \left(\tanh \left(\gamma_2 \dot{\theta} \right) - \tanh \left(\gamma_3 \dot{\theta} \right) \right) + \\ & \gamma_4 \tanh \left(\gamma_5 \dot{\theta} \right) + \gamma_6 \dot{\theta} - \gamma_1 \left(\tanh \left(\bar{\gamma}_2 \dot{\theta}_d \right) - \tanh \left(\bar{\gamma}_3 \dot{\theta}_d \right) \right) - \\ & \gamma_4 \tanh \left(\bar{\gamma}_5 \dot{\theta}_d \right) - \gamma_6 \dot{\theta}_d \end{aligned} \quad (201)$$

Now we take the time derivative and get

$$\begin{aligned} \dot{\bar{f}}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right) = & \ddot{\theta} \left(\gamma_1 \left(\gamma_2 \text{sech}^2 \left(\gamma_2 \dot{\theta} \right) - \right. \right. \\ & \left. \left. \gamma_3 \text{sech}^2 \left(\gamma_3 \dot{\theta} \right) \right) + \gamma_4 \gamma_5 \text{sech}^2 \left(\gamma_5 \dot{\theta} \right) + \gamma_6 \right) - \\ & \ddot{\theta}_d \left(\gamma_1 \left(\bar{\gamma}_2 \text{sech}^2 \left(\bar{\gamma}_2 \dot{\theta}_d \right) - \bar{\gamma}_3 \text{sech}^2 \left(\bar{\gamma}_3 \dot{\theta}_d \right) \right) + \right. \\ & \left. \gamma_4 \bar{\gamma}_5 \text{sech}^2 \left(\bar{\gamma}_5 \dot{\theta}_d \right) + \gamma_6 \right) \end{aligned} \quad (202)$$

We note that

$$0 \leq \text{sech}(x) \leq 1 \quad (203)$$

and that $\gamma_2 > \gamma_3$ for this to be a positive function and thus work as a friction model. Based on this knowledge, we can say

$$\left| \dot{\bar{f}}_f \left(\dot{\theta} \right) - \dot{f}_f \left(\dot{\theta}_d \right) \right| \leq \ddot{\theta} (\gamma_1 \gamma_2 - \gamma_1 \gamma_3 + \gamma_4 \gamma_5 + \gamma_6) - \ddot{\theta}_d (\gamma_1 \bar{\gamma}_2 - \gamma_1 \bar{\gamma}_3 + \gamma_4 \bar{\gamma}_5 + \gamma_6) \quad (204)$$

$$\leq (\gamma_1 \hat{\gamma}_2 - \gamma_1 \hat{\gamma}_3 + \gamma_4 \hat{\gamma}_5 + \gamma_6) * (r - \alpha_1 (e_2 - \alpha_1 e_1) - \alpha_2 e_2) \quad (205)$$

$$\leq c (r - \alpha_1 (e_2 - \alpha_1 e_1) - \alpha_2 e_2) \quad (206)$$

where $\hat{\gamma}_2 = \max \{ \gamma_2, \bar{\gamma}_2 \}$, $\hat{\gamma}_3 = \min \{ \gamma_3, \bar{\gamma}_3 \}$, and $\hat{\gamma}_5 = \max \{ \gamma_5, \bar{\gamma}_5 \}$.

Therefore, we can now say that:

$$\left| \tilde{N} \right| \leq c_1 r + c_2 e_2 + c_3 e_1 \quad (207)$$

$$\leq c_{max} \|\mathbf{z}\| \quad (208)$$

where

$$\begin{aligned} c_1 = & \left(\frac{1}{2} \dot{J}_{max} + \left| Y_d \Gamma \dot{Y}_d \right| + (J + J_{max}) |\alpha_1 - \alpha_2| + \right. \\ & \left. \dot{J}_{max} + c \right) \end{aligned} \quad (209)$$

$$c_2 = \left| 1 + \left(2\dot{J}_{max} + (J + J_{max})(\alpha_2 - \alpha_1) \right) \alpha_2 \right| + \left| J_{max}\alpha_1 + \ddot{J}_{max} + J\alpha_1^2 + \dot{J}_{max}\alpha_1 - (\alpha_1 + \alpha_2)c \right| \quad (210)$$

$$c_3 = \left(\ddot{J}_{max}\alpha_1 + 2\dot{J}_{max}\alpha_1^2 + \left(J + \dot{J}_{max} \right) \alpha_1^3 + \alpha_1^2 c \right) \quad (211)$$

and $c_{max} = \max\{c_1, c_2, c_3\}$. Note that the $Y_d\Gamma\dot{Y}_d$ term is bounded as it is a bounded function of the desired trajectory θ_d and its first three derivatives, which are bounded. \square

Lemma 2. *Given the following:*

$$E = \dot{\theta} \sin(\theta) e_\tau - \cos(\theta) \dot{e}_\tau \quad (212)$$

$$E_d = \dot{\theta}_d e_\tau \quad (213)$$

we can show that

$$\|\tilde{E}\| \leq \rho_E (\|\mathbf{z}\|) \|\mathbf{z}\| \quad (214)$$

where

$$\mathbf{z} = [e_1 \quad e_2 \quad r \quad e_\tau \quad e_J] \quad (215)$$

$$\tilde{E} = E - E_d \quad (216)$$

Proof. We begin by plugging E and E_d into the definition of \tilde{E} and applying the control law (49).

$$\tilde{E} = E - E_d \quad (217)$$

$$= \dot{\theta} \sin(\theta) e_\tau - \cos(\theta) \dot{e}_\tau - \dot{\theta}_d e_\tau \quad (218)$$

$$= \dot{\theta} \sin(\theta) e_\tau - \cos(\theta) (\dot{\tau}_{sd} - k_1 e_\tau - \dot{\tau}_s d) - \dot{\theta}_d e_\tau \quad (219)$$

$$= \dot{\theta} \sin(\theta) e_\tau + \cos(\theta) k_1 e_\tau - \dot{\theta}_d e_\tau \quad (220)$$

we can then show that

$$\tilde{E} \leq \dot{\theta} e_\tau + k_1 e_\tau - \dot{\theta}_d e_\tau \quad (221)$$

$$\leq \dot{\theta}_d e_\tau - \dot{e}_1 e_\tau + k_1 e_\tau - \dot{\theta}_d e_\tau \quad (222)$$

$$\leq (k_1 - e_2 + \alpha_1 e_1) e_\tau \quad (223)$$

$$\|\tilde{E}\| \leq (k_1 + (1 + \alpha_1) \|\mathbf{z}\|) \|\mathbf{z}\| \quad (224)$$

$$\leq \rho_E (\|\mathbf{z}\|) \|\mathbf{z}\| \quad (225)$$

where k_1 is the first term on the diagonal of K . \square

Lemma 3. *Given a function*

$$L(t) = r(N_d(t) - \beta \text{sgn}(e_2(t))) \quad (226)$$

If $\beta > \zeta_{N_d} + \frac{1}{\alpha_2} \zeta_{\dot{N}_d}$, then

$$\int_{t_0}^t L(\tau) d\tau \leq \beta \|e_2(t_0)\| - e_2(t_0) N_d(t_0) \quad (227)$$

Proof.

$$\int_{t_0}^t L(\tau) d\tau = \int_{t_0}^t r(N_d(\tau) - \beta \text{sgn}(e_2(\tau))) d\tau \quad (228)$$

$$= \int_{t_0}^t \dot{e}_2 N_d + \alpha_2 e_2 N_d - \dot{e}_2 \beta \text{sgn}(e_2) - \alpha_2 e_2 \beta \text{sgn}(e_2) d\tau \quad (229)$$

$$= \int_{t_0}^t \alpha_2 e_2 (N_d + \beta \text{sgn}(e_2)) d\tau + \int_{t_0}^t \dot{e}_2 N_d d\tau - \int_{t_0}^t \dot{e}_2 \beta \text{sgn}(e_2) d\tau \quad (230)$$

By integration by parts

$$\int_{t_0}^t L(\tau) d\tau = \int_{t_0}^t \alpha_2 e_2 (N_d + \beta \text{sgn}(e_2)) d\tau + e_2 N_d|_{t_0}^t - \int_{t_0}^t e_2 \dot{N}_d d\tau - \quad (231)$$

$$\int_{t_0}^t \dot{e}_2 \beta \text{sgn}(e_2) d\tau = \int_{t_0}^t \alpha_2 e_2 \left(N_d + \frac{1}{\alpha_2} \dot{N}_d - \beta \text{sgn}(e_2) \right) d\tau + e_2(t) N_d(t) - e_2(t_0) N_d(t_0) - \beta |e_2(t)| + \beta |e_2(t_0)| \quad (232)$$

$$\leq \int_{t_0}^t \alpha_2 |e_2| \left(|N_d| + \frac{1}{\alpha_2} |\dot{N}_d| - \beta \right) d\tau + (|N_d(t)| - \beta) |e_2(t)| + \beta |e_2(t_0)| - e_2(t_0) N_d(t_0) \quad (233)$$

From here we can see that if β satisfies the given conditions, (227) holds. This Lemma and proof are restated in this appendix, it is taken from [14] and [13]. \square

Lemma 4. *Given the constraints on the abstract state given in (36) and (37), the PD control law given in (39), and the swarm control law given in (49), $|\dot{J}_s| < \dot{J}_{max}$ on some domain $D_{J_{max}}$.*

Proof. We start with the definition of e_J from (50), substitute (36) and solve for J_s

$$J_s = F_1 \tau_{sd}^2 + F_2 - e_J \quad (234)$$

We then take the time derivative to get

$$\dot{J}_s = 2F_1 \tau_{sd} \dot{\tau}_{sd} - \dot{e}_J \quad (235)$$

We substitute in the time derivative of (39) and the swarm control law (49)

$$\dot{J}_s = 2F_1 \tau_{sd} \left(k_1 \dot{\theta} + k_2 \ddot{\theta} \right) - K_{s2} e_J \quad (236)$$

where K_1 and K_2 are the elements of $K_p d$ and K_{s2} is the second diagonal element of the K matrix from the swarm controller.

We now substitute the equations of motion for the parent system (25)

$$\dot{J}_s = 2F_1 \tau_{sd} \left(k_1 \dot{\theta} + \frac{k_2}{J + F_1 \tau_{sd}^2 + F_2 - e_J} \left(-\cos(\theta) \tau_{sd} - \dot{J}_s \dot{\theta} - f_f(\dot{\theta}) \right) \right) - K_{s2} e_J \quad (237)$$

Now, we solve for \dot{J}_s and substitute the parent system control law (39).

$$\dot{J}_s = \frac{E_1 E_2 - E_3 E_4}{E_5} \quad (238)$$

where

$$E_1 = J + F_1 \left(k_1 \theta + k_2 \dot{\theta} \right)^2 + F_2 - e_J \quad (239)$$

$$E_2 = 2F_1 k_2 \dot{\theta} \left(k_1 \theta + k_2 \dot{\theta} \right) - K_{s2} e_J \quad (240)$$

$$E_3 = 2F_1 k_2 \left(k_1 \theta + k_2 \dot{\theta} \right) \quad (241)$$

$$E_4 = \cos(\theta) \left(k_1 \theta + k_2 \dot{\theta} \right) + f_f \left(\dot{\theta} \right) \quad (242)$$

$$E_5 = E_1 + 2F_1 k_2 \dot{\theta} \left(k_1 \theta + k_2 \dot{\theta} \right) \quad (243)$$

We can now find $\left| \dot{J}_s \right|$ and bound it with $\|\mathbf{z}\|$.

$$\left| \dot{J}_s \right| \leq \frac{|E_1 E_2| + |E_2 E_4|}{|E_5|} \quad (244)$$

where

$$\begin{aligned} |E_1 E_2| \leq & 2F_2 (k_1 + k_2)^2 (k_1^2 + k_2^2) \|\mathbf{z}\|^4 + \\ & (F_1 (k_1 + k_2) K_{s2} + 2F_1 (k_1^2 + k_2^2)) \|\mathbf{z}\|^3 + \\ & ((J + F_2) 2F_1 (k_1^2 + k_2^2) + K_{s2}) \|\mathbf{z}\|^2 + \\ & (J + F_2) K_{s2} \|\mathbf{z}\| \end{aligned} \quad (245)$$

$$\begin{aligned} |E_3 E_4| \leq & 2F_1 (k_1 + k_2) (k_1 + k_2^2) \|\mathbf{z}\|^2 + \\ & 2F_1 (k_1 + k_2^2) \|\mathbf{z}\| \left| f_f \left(\dot{\theta} \right) \right| \end{aligned} \quad (246)$$

$$\begin{aligned} |E_5| \leq & \left(F_1 (k_1 + k_2)^2 + 2F_1 (k_1 + k_2^2) \right) \|\mathbf{z}\|^2 + \\ & \|\mathbf{z}\| + J + F_2 \end{aligned} \quad (247)$$

Given the friction model in (42), we can see that

$$\left| f_f \left(\dot{\theta} \right) \right| \leq \gamma_4 + \gamma_6 \left| \dot{\theta} \right| \leq \gamma_4 + \gamma_6 \|\mathbf{z}\| \quad (248)$$

We can substitute this into (246)

$$\begin{aligned} |E_3 E_4| \leq & 2F_1 (k_1 + k_2^2) (k_1 + k_2 + \gamma_6) \|\mathbf{z}\|^2 + \\ & 2F_1 \gamma_4 (k_1 + k_2^2) \|\mathbf{z}\| \end{aligned} \quad (249)$$

We can now show that

$$\left| \dot{J}_s \right| \leq \|\mathbf{z}\| \rho_J (\|\mathbf{z}\|) \quad (250)$$

where

$$\rho_J (\|\mathbf{z}\|) = \frac{\alpha_1 \|\mathbf{z}\|^3 + \alpha_2 \|\mathbf{z}\|^2 + \alpha_3 \|\mathbf{z}\| + \alpha_4}{\beta_1 \|\mathbf{z}\|^2 + \|\mathbf{z}\| + \beta_3} \quad (251)$$

$$\alpha_1 = 2F_2^2 (k_1 + k_2)^2 (k_1^2 + k_2^2) \quad (252)$$

$$\alpha_2 = F_1 (k_1 + k_2) K_{s2} + 2F_1 (k_1^2 + k_2^2) \quad (253)$$

$$\begin{aligned} \alpha_3 = & (J + F_2) 2F_1 (k_1^2 + k_2^2) + K_{s2} + \\ & 2F_1 (k_1 + k_2^2) (k_1 + k_2 + \gamma_6) \end{aligned} \quad (254)$$

$$\alpha_4 = (J + F_2) K_{s2} + 2F_1 \gamma_4 (k_1 + k_2^2) \quad (255)$$

$$\beta_1 = F_1 (k_1 + k_2)^2 + 2F_1 (k_1 + k_2^2) \quad (256)$$

$$\beta_2 = J + F_2 \quad (257)$$

We can now show that $\dot{J}_s \leq \dot{J}_{max}$ on some domain D_{Jmax} where

$$J_{max} = \max \{ \|\mathbf{z}\| \rho_J (\|\mathbf{z}\|) \mid \forall \mathbf{z} \in D_{Jmax} \} \quad (258)$$

□

REFERENCES

- [1] Y. Tan and Z.-y. Zheng, "Research advance in swarm robotics," *Defence Technology*, vol. 9, no. 1, pp. 18–39, 2013.
- [2] G. Habibi, Z. Kingston, W. Xie, M. Jellins, and J. McLurkin, "Distributed centroid estimation and motion controllers for collective transport by multi-robot systems," in *Robotics and Automation (ICRA), 2015 IEEE International Conference on*. IEEE, 2015, pp. 1282–1288.
- [3] G. A. Pereira, M. F. Campos, and V. Kumar, "Decentralized algorithms for multi-robot manipulation via caging," *The International Journal of Robotics Research*, vol. 23, no. 7-8, pp. 783–795, 2004.
- [4] J. Fink, M. A. Hsieh, and V. Kumar, "Multi-robot manipulation via caging in environments with obstacles," in *Robotics and Automation, 2008. ICRA 2008. IEEE International Conference on*. IEEE, 2008, pp. 1471–1476.
- [5] R. Gross and M. Dorigo, "Towards group transport by swarms of robots," *International Journal of Bio-Inspired Computation*, vol. 1, no. 1-2, pp. 1–13, 2009.
- [6] F. A. Goodarzi and T. Lee, "Dynamics and control of quadrotor uavs transporting a rigid body connected via flexible cables," in *American Control Conference (ACC), 2015*. IEEE, 2015, pp. 4677–4682.
- [7] K. Klausen, T. I. Fossen, T. A. Johansen, and A. P. Aguiar, "Cooperative path-following for multirotor uavs with a suspended payload," in *Control Applications (CCA), 2015 IEEE Conference on*. IEEE, 2015, pp. 1354–1360.
- [8] C. J. Blower, W. Lee, and A. M. Wickenheiser, "The development of a closed-loop flight controller with panel method integration for gust alleviation using biomimetic feathers on aircraft wings," in *SPIE Smart Structures and Materials+ Nondestructive Evaluation and Health Monitoring*. International Society for Optics and Photonics, 2012, pp. 83390I–83390I.
- [9] M. Boberg, G. Feltrin, and A. Martinoli, "A novel bridge section model endowed with actively controlled flap arrays mitigating wind impact," in *Robotics and Automation (ICRA), 2015 IEEE International Conference on*. IEEE, 2015, pp. 1837–1842.
- [10] K. L. Crandall, C. Whitehead, S. Dong, and A. Wickenheiser, "Using abstraction for swarm control of a parent system," in *2016 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2016, pp. 5344–5349.
- [11] C. Belta and V. Kumar, "Abstraction and control for groups of robots," *Robotics, IEEE Transactions on*, vol. 20, no. 5, pp. 865–875, 2004.
- [12] H. K. Khalil, *Nonlinear Systems*, 3rd ed. New Jersey: Prentice Hall, Inc., 2002.
- [13] B. Xian, D. M. Dawson, M. S. de Queiroz, and J. Chen, "A continuous asymptotic tracking control strategy for uncertain nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1206–1211, 2004.
- [14] P. M. Patre, W. MacKunis, C. Makkar, and W. E. Dixon, "Asymptotic tracking for systems with structured and unstructured uncertainties," in *Decision and Control, 2006 45th IEEE Conference on*. IEEE, 2006, pp. 441–446.
- [15] P. M. Patre, W. MacKunis, K. Kaiser, and W. E. Dixon, "Asymptotic tracking for uncertain dynamic systems via a multilayer neural network feedforward and rise feedback control structure," *Automatic Control, IEEE Transactions on*, vol. 53, no. 9, pp. 2180–2185, 2008.
- [16] G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, 6th ed. Pearson, 2010, ch. 3, pp. 132–134.



Kyle L. Crandall (M'14) received the B.S. and M.S. degrees in mechanical engineering, with a focus on robotics, in 2015 from the University of Utah, Salt Lake City. He is currently a Ph.D. candidate at The George Washington University, Washington, DC. His research interests are in control of multi-agent systems. He is a student member of the IEEE.



Adam M. Wickenheiser (M'08) received the B.S. degree in mechanical engineering (with a minor in applied mathematics) in 2002 and the M.S. and Ph.D. degrees in aerospace engineering from Cornell University, Ithaca, in 2006 and 2008, respectively.

Since 2010, he has been in the Faculty of the Department of Mechanical & Aerospace Engineering at the George Washington University, Washington, DC, where he is currently an Assistant Professor. From 2008-2009, he was a postdoctoral associate with the Sibley School of Mechanical & Aerospace

Engineering at Cornell University. His current research interests include bio-inspired flight, multi-functional materials and systems, and energy harvesting for autonomous systems.

Prof. Wickenheiser has served as the Chair of the Energy Harvesting Technical Committee of the American Society of Mechanical Engineers (ASME) from 2014-2016, and is presently a member of the International Organizing Committee for the International Conference on Adaptive Structures and Technologies (ICAST). He was the recipient of the 2011 Intelligence Community Young Investigator Award.